3.2 Exponential growth and decay: Constant percentage rates

Learning Objectives:

- Understand exponential functions and consequences of constant percentage change.
- Calculate exponential growth, exponential decay, and the half-life.

3.2 Exponential growth and decay: Constant percentage rates

- An **exponential function** is a function that changes at a constant percentage rate.

- **Example:** If a population triples each hour, does this represent constant percentage growth?

If so, what is the percentage increase each hour?

Is the population size an exponential function of time?

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- **Solution:** The population changes each hour:

$$\text{Population next hour} = 3 \times \text{Current population}$$

Suppose we start with 100 individuals:

$$\text{Initial population} \ = 100$$

$$\text{Population after 1 hour} \ = 3 \times 100 = 300$$

$$\text{Population after 2 hours} = 3 \times 300 = 900$$

Let's look at this in terms of growth:

Growth over 1st hour $= 300 - 100 = 200 = 200\%$ increase over 100

Growth over 2nd hour $= 900 - 300 = 600 = 200\%$ increase over 300

The population is growing at a constant percentage rate, 200% each hour. Thus, the population is an exponential function of time.

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- **Exponential Formulas**

❑ The formula for an exponential function *y* of *t* is:

$$y = \text{Initial value } \times \text{Base}^{t}$$

❑ An exponential function *y* of *t* is characterized by the following property: When *t* increases by 1, to find the new value of *y*, we multiply the current value by the base.

$$y - \text{value for } t + 1 = \text{Base } \times \text{y} - \text{value for } t$$

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- **Example:** The value of a certain investment grows according to the rule:

$$\text{Next year's balance} = 1.07 \times \text{Current balance}$$

1. Find the percentage increase each year, and explain why the balance is an exponential function of time.
2. Assume that the original investment is \$800. Find an exponential formula that gives the balance in terms of time.
3. What is the balance after 10 years?

- **Solution:**

1. The next year's balance $= 1.07 \times$ this year's balance.

The next year's balance is 107% of this year's balance.

That is an increase of 7% per year. Because the balance grows by the same percentage each year, it is an exponential function of time.

2. Let $B =$ the balance in dollars after t years.

$$B = \text{Initial value } \times \text{Base}^t$$

The initial value $= \$800$. The base $= 1.07$. This gives the formula:

$$B = 800 \times 1.07^t$$

3. Balance after 10 years $= 800 \times 1.07^{10} = \$1573.72$.

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- **Example:** Consider the investment from the previous example where the balance *B* after *t* years is given by:

$$B = 800 \times 1.07^t \text{ dollars}$$

What is the growth of the balance over the first 10 years? Compare this with the growth from year 40 to year 50.

- **Solution:** The balance after 10 years was $1573.72.

$$\text{Growth over first 10 years} = \$1573.72 - 800 = \$773.72.$$

To calculate the growth from year 40 to year 50:

$$\text{Balance after 40 years} = 800 \times 1.07^{40} = \$11{,}979.57$$
$$\text{Balance after 50 years} = 800 \times 1.07^{50} = \$23{,}565.62$$

That is an increase of $\$23{,}565.62 - \$11{,}979.57 = \$11{,}586.05$. That is almost 15 times the growth over the first 10 years.

3.2 Exponential growth and decay: Constant percentage rates

- **Exponential Growth**

1. A quantity grows exponentially when it increases by a constant percentage over a given period.
2. If r is the percentage growth per period then the base of the exponential function is $1+\mathrm{r}$. For exponential growth, the base is always greater than 1.

$$\text{Amount} = \text{Initial value } \times (1+r)^{\mathrm{t}}$$

Here, t is the number of periods.

3. Typically, exponential growth starts slowly and then increases rapidly.

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- **Example:** U.S. health-care expenditures in 2010 reached 2.47 trillion dollars. In the near term this is expected to grow by 6.5% each year. Assuming that this growth rate continues, find a formula that gives health-care expenditures as a function of time. If this trend continues, what will health-care expenditures be in 2030?

- **Solution:** Let *H* be the expenditures *t* years after 2010.

$$H = \text{Initial value } \times (1 + r)^t = 2.47 \times (1 + 0.065)^t.$$

To predict health-care expenditures in 2030, use $t = 20$ in the formula for *H*:

$$\text{Expenditures in } 2030 = 2.47 \times (1 + 0.065)^{20} \text{ trillion dollars.}$$

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- **Solut**

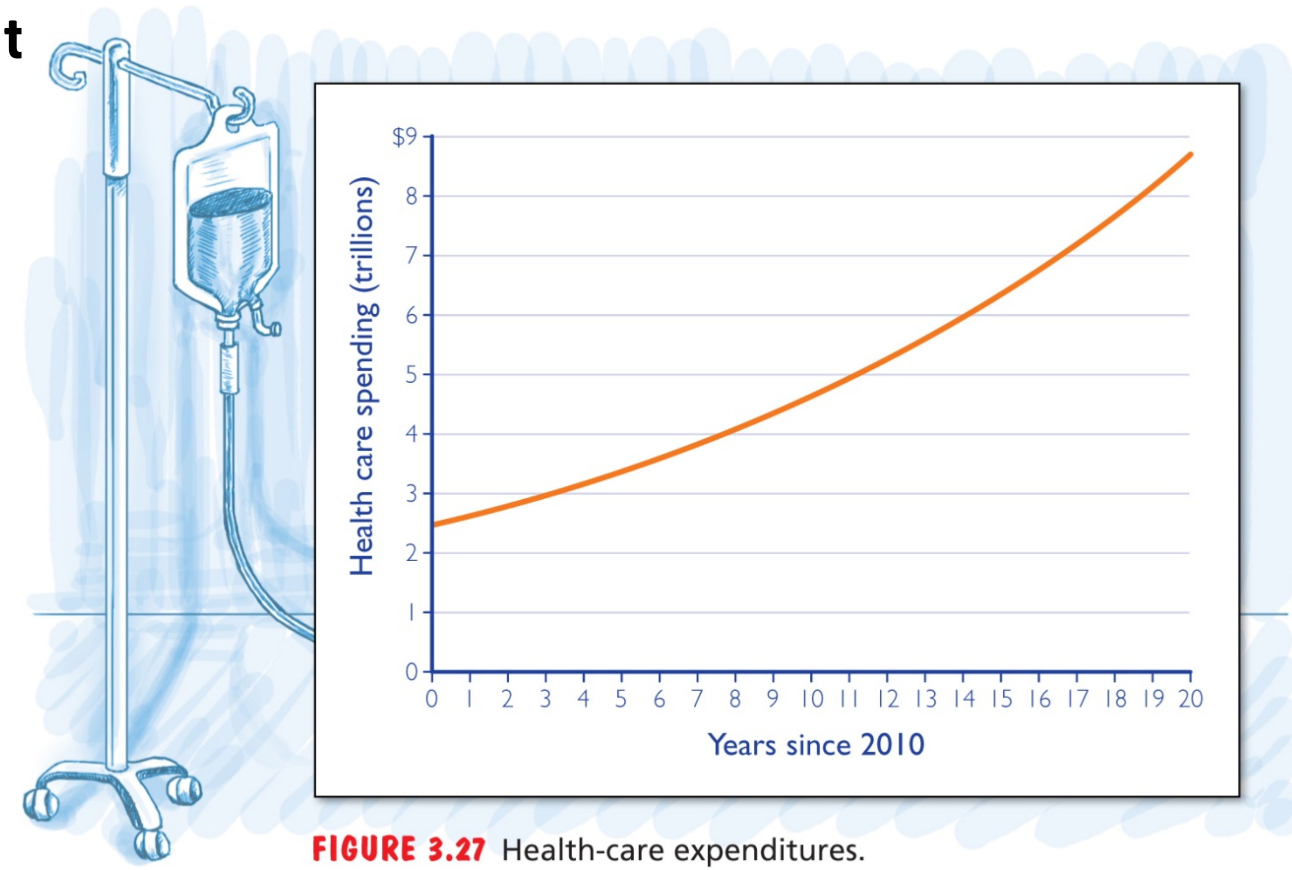

FIGURE 3.27 Health-care expenditures.

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- **Exponential Decay**

1. A quantity decays exponentially when it decreases by a constant percentage over a given period.
2. If r is the percentage decay per period, then the base of the exponential function is $1 - r$. For exponential decay, the base is

$$\text{Amount} = \text{Initial value} \times (1 - r)^t$$

   Here, t is the number of periods.

3. Typically, exponential decay is rapid first but eventually slows.

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- **Example:** After antibiotics are administered, the concentration in the bloodstream declines over time. Suppose that 70 milligrams (mg) of amoxicillin are injected and that the amount of the drug in the bloodstream declines by 49% each hour.

Find an exponential formula that gives the amount of amoxicillin in the bloodstream as a function of time since the injection. Another injection will be required when the level declines to 10 mg. Will another injection be required before five hours?

- **Solution:** Let *A* be the amount of amoxicillin in the bloodstream after *t* hours. The base of the exponential function is:

$$1 - r = 1 - 0.49 = 0.51$$

The initial value = 70 mg:

$$A = \text{Initial value } \times (1 - r)^t = 70 \times 0.51^t$$

To find the amount of amoxicillin after 5 hours:

$$A = 70 \times 0.51^5 = 2.4 \text{ mg}$$

The result is about 2.4 mg, which is less than the minimum of 10 mg. Thus, another injection will be needed before 5 hours.

- **Solution:**

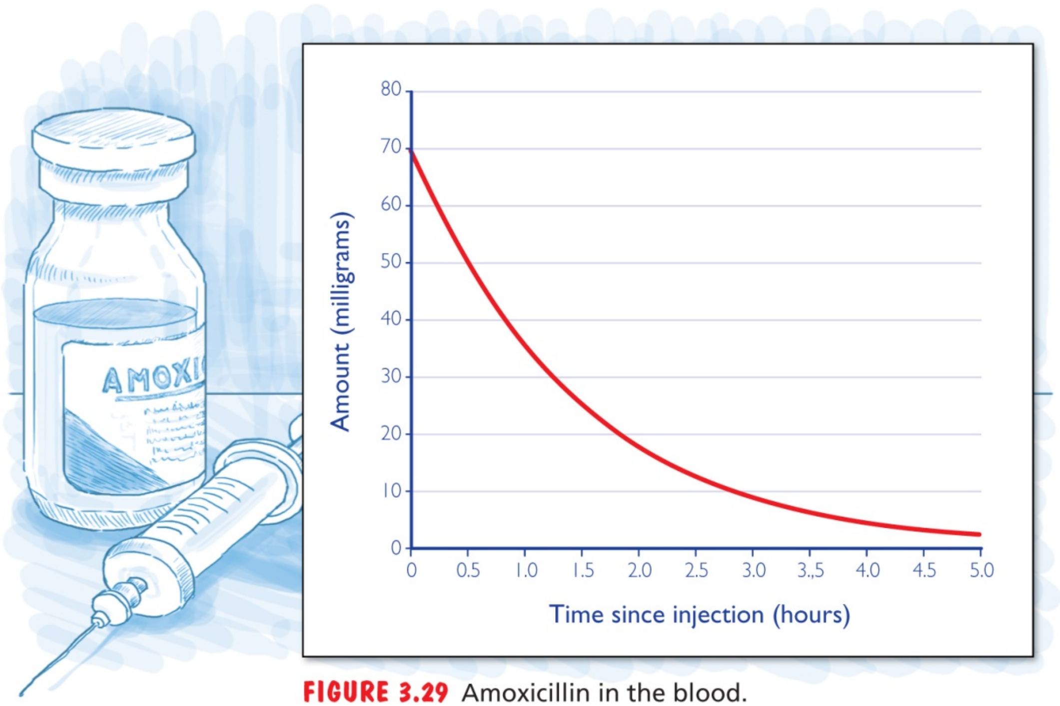

FIGURE 3.29 Amoxicillin in the blood.

The graph shows that the level declines to 10 mg in about 3 hours.

- The **half-life** of a radioactive substance is the time it takes for half of the substance to decay.
- After *h* half-lives, the amount of a radioactive substance rem

Half-life formula

$$\text{Amount remaining} = \text{Initial amount } \times \left(\frac{1}{2}\right)^{t}$$

- We can find the amount remaining after *t* years by first expressing *t* in terms of half-lives and then using the formula above.

- **Example:** The carbon-14 in the organism decays, with a half-life of 5770 years. Suppose a tree contains C_0 grams of carbon-14 when it was cut down. What percentage of the original amount of carbon-14 would we find if it was cut down 30,000 years ago?
- **Solution**:

$$\text{The amount } C \text{ remaining after } h \text{ half} - \text{life} = C_{\text{o}} \times \left(\frac{1}{2}\right)^{h}$$

$$5770 = 1 \text{ half-life} \implies 30{,}000 \text{ years}/5770 = 5.20 \text{ half-lives}$$

$$\text{Amount after } 30{,}000 \text{ years} = C_{\text{o}} \times \left(\tfrac{1}{2}\right)^{5.20} = 0.027 C_{\text{o}} \text{ grams.}$$

Thus, about 2.7% of the original amount of carbon-14 remains after 30,000 years.