5.3 Counting and theoretical probabilities: How many?

Learning Objectives:

- Learn how to count outcomes without listing all of them:
  - Simple counting
  - The Counting Principle
  - Applying counting to probabilities
  - Independent events

5.3 Counting and theoretical probabilities: How many?

- **Simple counting:** Consider counting the number of outcomes in the case of flipping two coins. We distinguish the two coins (a nickel and a dime) and then simply list the four possible outcomes: **HH, HT, TH, TT.**

If we are flipping three coins (a penny, a nickel, and a dime), then there are eight possible outcomes: for each of four possibilities for the first two coins, there are two possibilities for the third coin.

The total number of outcomes $= 4 \times 2 = 2^3 = 8$.

Penny	Nickel	Dime
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

5.3 Counting and theoretical probabilities: How many?

- **Example:** How many possible outcomes are there if we toss 12 coins?

- **Solution:** Reasoning as above, we see that each extra coin doubles the number of outcomes.

  A total of $2^{12} = 4096$ possible outcomes.

5.3 Counting and theoretical probabilities: How many?

- The **counting principle** tells us how to calculate the results of two experiments performed in succession.

  Suppose there are N outcomes for the first experiment. If for each outcome of the first experiment, there are M outcomes of the second experiment, then the number of possible outcomes for the two experiments is $N \times M$.

  This principle extends to any number of such experiments.

5.3 Counting and theoretical probabilities: How many?

- **Example:** How many three-letter codes can we make if the first letter is a vowel (A, E, I, O, or U)? One such code is OXJ.

- **Solution:** We think of filing in three banks:

  $$\_\ \_\ \_$$

  There are 5 letters that can go in the first blank and 26 in each of the other two:

  $$\underline{5}\ \ \underline{26}\ \ \underline{26}$$

  Applying the Counting Principle, we find a total of

  $$5 \times 26 \times 26 = 3380 \text{ codes.}$$

- **Example:** Automobile license plates in the state of Nevada typically consist of three numerals followed by three letters of the alphabet, such as 072 ZXE.

1. How many such license plates are possible?
2. How many such plates are possible if we insist that on each plate no numeral can appear more than once and no letter can appear more than once?

- **Solution:**

1. Three slots for numbers and three slots for letters:

   _ _ _   _ _ _
   Numbers   letters

   For each number slot, there are 10 possible numerals (0 through 9) to use: $\underline{10}\ \underline{10}\ \underline{10}$ _ _ _
   Numbers   letters

   For each letter slot, there are 26 possible letters of alphabet:

   $\underline{10}\ \underline{10}\ \underline{10}\ \ \underline{26}\ \underline{26}\ \underline{26}$
   Numbers   letters

   Applying the Counting Principle gives

   $10 \times 10 \times 10 \times 26 \times 26 \times 26 = 17{,}576{,}000$ ways.

- **Solution (cont.):**

2. For the first number slot, we have 10 choices. For the second slot, we can't repeat the number in the first blank, so we have only 9 choices. For the third slot, we can't repeat the numbers used in the first two slots, so we have a choice of only 8 numbers:

$$\underline{10}\ \underline{9}\ \underline{8}\quad \_\ \_\ \_$$

Numbers letters

We fill in the blank for the letters in same the manner:

$$\underline{10}\ \underline{9}\ \underline{8}\quad \underline{26}\ \underline{25}\ \underline{24}$$

Numbers letters

$$\text{Number of plates} = 10 \times 9 \times 8 \times 26 \times 25 \times 24 = 11{,}232{,}000.$$

5.3 Counting and theoretical probabilities: How many?

- **Example:** Suppose you draw a card, put it back in the deck, and draw another. What is the probability that the first card is an ace and the second one is a jack?
- **Solution:** The number of ways we can draw two cards

= the total number of possible outcomes

The number of ways we can draw an ace followed by a jack

= the number of favorable outcomes

There are 52 possible cards to place in the first blank, and there are 52 possible cards to place in the second blank.

Using the Counting Principle:

$$52 \times 52 = 2704 \text{ ways}$$

This is the total number of possible outcomes.

- **Solution (cont.):** Now, how many ways there are to fill these two blanks with an ace followed by a jack.

There are four aces in the deck, so there are four choices to fill the first blank: $\underline{4}$ _

After this has been done, there are four jacks with which to fill the second blank: $\underline{4}\ \underline{4}$

Again, using the Counting Principle: $4 \times 4 = 16$ ways to fill the two blanks:

$$\text{P(Ace followed by jack)} = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{16}{2704}$$

$$= 0.006 = 0.6\%$$

5.3 Counting and theoretical probabilities: How many?

- **Example:** Suppose there are 10 people willing to serve as officer for a club. It's decided just to put the 10 names in a hat and draw three of them out in succession. The first name drawn is declared president, the second name vice president, and the third name treasurer.

1. How many possible election outcomes are there?
2. John is a candidate. What is the probability that he will be vice president?
3. Mary and Jim are also candidates along with John. What is the probability that all three will be selected to office?
4. What is the probability that none of the three (Mary, Jim, and John) will be selected to office?
5. What is the probability that at least one of the three (Mary, Jim, and John) will **not** be selected?

- **Solution:**

1. Imagine that we have three blanks corresponding to president, vice president, and treasure:

$$\overline{\text{P}} \quad \overline{\text{VP}} \quad \overline{\text{T}}$$

There are 10 possibilities for the president blank. When that is filled, there are nine names left, so there are nine possibilities for the vice president blank. When that is filled, there are eight possibilities for the third blank:

$$\frac{10}{\text{P}} \quad \frac{9}{\text{VP}} \quad \frac{8}{\text{T}}$$

The Counting Principle gives:

$$\text{Number of outcomes} = 10 \times 9 \times 8 = 720$$

- **Solution (cont.):**

2. We must count the number of ways John can be vice president. In this case, John is not president, so there are nine possible names for the president blank. There is just one possibility, John, for the vice president blank, and that leaves eight possibilities for the treasurer blank:

$$\frac{9}{\text{P}} \quad \frac{1}{\text{VP}} \quad \frac{8}{\text{T}}$$

$$\text{Number of ways John is vice president} = 9 \times 1 \times 8 = 72.$$

This is the number of favorable outcomes.

$$\text{Probability John is vice president} = \frac{\text{Favorable outcomes}}{\text{Total outcomes}}$$

$$= \frac{72}{720} = 0.1 = 10\%$$

- **Solution (cont.):**

3. We want to arrange the three names John, Jim, and Mary in the three office-holder blanks. There are three possible names for the first, two for the second, and one for third:

$$\frac{3}{\mathrm{P}} \quad \frac{2}{\mathrm{VP}} \quad \frac{1}{\mathrm{T}}$$

Thus, the total number of ways three candidates can all win office is $3 \times 2 \times 1 = 6$. These are the favorable outcomes.

$$\text{Probability all three selected} = \frac{6}{720} = 0.008 = 0.8\%$$

- **Solution (cont.):**

4. If these three names don't appear in the office blanks, then there are seven possibilities for the first, six for the second, and five for the third:

$$\frac{7}{\text{P}} \quad \frac{6}{\text{VP}} \quad \frac{5}{\text{T}}$$

That is a total of $7 \times 6 \times 5 = 210$ favorable outcomes.

$$\text{Probability none selected} = \frac{210}{720} = 0.292 = 29.2\%$$

- **Solution (cont.):**

5. We found in part 3 that the probability that all three are selected is 6/720.

$$\text{Probability at least one } \textbf{not} \text{ selected}$$

$$= \text{Probability of all three selected does } \textbf{not} \text{ occur}$$

$$= 1 - \text{Probability all three selected}$$

$$= 1 - \frac{6}{720} = 0.992 = 99.2\%$$

- Two events are **Independent** if knowing that one event occurs has no effect on the probability of the occurrence of the other.

<u>Product formula for Independent Events</u>

$$P(\text{A and B}) = P\,(A) \times P(B)$$

- **Example:** Suppose that 1 in 500 digital cameras is defective and 3 in 1000 printers are defective. On a shopping trip, I purchase a digital camera and a printer. What is the probability that both the camera and the printer are defective?
- **Solution:** $P(\text{Defective camera and Decfective Printer})$

$$= \frac{1}{500} \times \frac{3}{1000} = \frac{3}{500{,}000} = \frac{6}{1 \text{ million}}$$

Learning Objectives:

- Effective counting techniques apply in many practical settings:
  - Permutations
  - More on arrangements
  - Calculations using factorials
  - Combinations
  - Hand calculation of combinations
  - Probabilities with permutations or combinations

- A **permutation** of items is an arrangement of the items in a certain order. Each item can be used only in the sequence.

Permutations of *n* items

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

- **Example:** In how many ways can I arrange the four letters A, B, C, D using each letter **only once**?

- **Solution:** The number of permutations of four letters is:

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

5.4 More ways of counting: Permuting and combining

- The number of **permutations of *n* items taken *k* at a time:** The number of ways to select *k* items from *n* distinct items and arrange them in order.
- **Example:** In how many ways can we select 3 people from a group of 10 and award them first, second, and third prizes? (This is the number of permutations of 10 items taken 3 at a time.)
- **Solution:** We think of filling 3 slots with names selected from the 10 people: $\_\ \_\ \_$

  There are 10 names available for the first slot. Nobody gets more than one prize, there are only 9 available for the second, and then 8 available for the third: $\underline{10}\ \underline{9}\ \underline{8}$

  The Counting Principle shows $10 \times 9 \times 8 = 720$ ways to select the winners.

5.4 More ways of counting: Permuting and combining

- **Example:** A political organization wants to observe the voting procedures at five polling places. The organization has five poll watchers available.

1. In how many different ways can the poll watchers be assigned?
2. For each of the polling places, one of the available watchers lives in the precinct for that polling place. If the watchers are assigned at random, what is the probability that each one will be assigned to the polling place for the precinct where he or she lives?
3. Suppose now that the organization wants to observe only two of the polling places. In how many ways can the organization assign these places to two of the available watchers?
4. Assume now that there are 20 polling places and 20 poll watchers available. How many permutations are there of the 20 watchers?

- **Solution:**

1. The number of ways to arrange five polling places is:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

2. From part 1, there are 120 outcomes:

$$\text{Probability assigned to his/her home precinct} = \frac{1}{120} = 0.008$$

3. There are five choices to watch the first polling place and four choices to watch the second polling place.

$$5 \times 4 = 20 \text{ different arrangements}$$

4. The number of arrangements of 20 items:

$$20! = 2{,}432{,}902{,}008{,}176{,}640{,}000$$

Generally speaking, specialized mathematical software is needed to handle numbers of this size, and hand calculation is out of the question.

Permutations of *n* Items Taken *k* at a Time

$$\text{Permutations of } n \text{ items taken } k \text{ at a time} = \frac{n!}{(n-k)!}$$

- **Example:** The producer of a talent show has seven slots to fill. Twenty-five acts have requested a slot. Use the permutations formula to find the number of ways to select seven acts and arrange them in a sequence.
- **Solution:** We are selecting 7 of 25 acts, so we use the permutations formula for 25 items taken 7 at a time:

$$\text{Permutations of 25 items taken 7 at a time} = \frac{25!}{(25-7)!} = \frac{25!}{18!}$$

- A **combination** of a group of items is a selection from that group in which **order is not taken** into account. No item can be used more than once in a combination.

Combinations

$$\text{Combinations of } n \text{ items taken } k \text{ at a time} = \frac{n!}{k!\,(n-k)!}$$

- **Example:** Use the combinations formula to express the number of three-person committees I can select from a group of six people.
- **Solution:** The number of combinations of six people taken three at a time: $n = 6,\ k = 3.$

$$\text{Combinations of 6 people taken 3 at a time} = \frac{6!}{3!\,(6-3)!} = \frac{6!}{3!\,3!}$$

- **Example:** Calculate by hand the number of combinations of nine items taken five at a time.

- **Solution:**

$$\text{Combinations of 9 items taken 5 at a time} = \frac{9!}{5!\,(9-5)!} = \frac{9!}{5!\,4!}$$

$$\frac{9!}{5!\,4!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(5\times4\times3\times2\times1)\times(4\times3\times2\times1)}$$

$$= \frac{9\times8\times7\times6\times\cancel{5\times4\times3\times2\times1}}{\cancel{5\times4\times3\times2\times1}\times4\times3\times2\times1}$$

$$= \frac{9\times\cancel{8}\times7\times6}{\cancel{4}\times3\times\cancel{2}\times1} = \frac{{}^{3}\cancel{9}\times7\times6}{\cancel{3}\times1} = \frac{3\times7\times6}{1} = 126$$

5.4 More ways of counting: Permuting and combining

- **Example:** *The New York Times* ran an article on June 2002 with the title, "Court That Ruled on Pledge Often Runs Afoul of Justices." The court in question is the Ninth Circuit Court, which ruled in 2002 that the Pledge of Allegiance to the flag is unconstitutional because it includes the phrase "under God." The article discusses the effect of having a large number of judges, and it states, "The judges have chambers throughout the circuit and meet only rarely. Assuming there are 28 judges, there are more than 3000 possible combinations of three-judge panels."

1. Is the article correct in stating that there are more than 3000 possible combinations consisting of three-judge panels? Exactly how many three-judge panels can be formed from the 28-judge court?
2. The article says that the judges meet rarely. Assume that there are 28 judges. How many 28-judge panels are there?

- **Solution:**

1. Combinations of 28 items taken 3 at a time:

$$= \frac{28!}{3!\,(28-3)!} = \frac{28!}{3!\,25!}$$

$$= \frac{28 \times 27 \times 26 \times \cancel{25 \times 24 \times 23 \times \cdots \times 1}}{(3 \times 2 \times 1) \times (\cancel{25 \times 24 \times 23 \times \cdots \times 1})}$$

$$= \frac{28 \times 27 \times 26}{3 \times 2 \times 1}$$

$$= 3276$$

The article is correct in stating that are more than 3000 possible three-judge panels. There are 3276 such combinations.

- **Solution (cont.):**

2. We are really being asked, "In how many ways can we choose 28 judges from 28 judges to form a panel?"

   Obviously only one way: All 28 judges go on the panel.

   We use the combinations formula with $n = 28$ and $k = 28$:

$$\text{Combinations of 28 items taken 28 at a time}$$
$$= \frac{28!}{28!\,(28-28)!} = \frac{28!}{28!\;\,0!} = 1$$

   This is the same as the answer we found before.

   Remember that $0! = 1$, and this formula is one reason why.

- **Example:** In a group of six men and four women, I select a committee of three at random. What is the probability that all three committee numbers are women?
- **Solution:** The number of ways to select a three-women committee from four women:

$$\text{Combinations of 4 items taken 3 at a time} = \frac{4!}{3!\,(4-3)!} = 4$$

The number of ways to select a three-person committee from the 10 people (the total number of outcomes):

$$\text{Combinations of 10 items taken 3 at a time} = \frac{10!}{3!\,(10-3)!} = 120$$

$$\text{Probability of all} - \text{female committee}$$

$$= \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{4}{120} = \frac{1}{30} = 0.03$$

5.4 More ways of counting: Permuting and combining

- **Example:** There are many different kinds of poker games, but in most of them the winner is ultimately decided by the best five-card hand. Suppose you draw five cards from a full deck.

1. How many five-card hands are possible?
2. What is the probability that exactly two of the cards are kings?
3. What is the probability that all of them are hearts?

Various poker hands.

- **Solution:**

1. The total number of hands:

$$\text{Combinations of 52 items taken 5 at a time} = \frac{52!}{5!\,(52-5)!}$$
$$= 2{,}598{,}960$$

2. Choose two from four kings:

$$\text{Combinations of 4 items taken 2 at a time} = \frac{4!}{4!\,(4-2)!} = 6$$

Choose 3 from not kings (48 cards are not kings)

$$\text{Combinations of 48 items taken 3 at a time} = \frac{48!}{3!\,(48-3)!}$$
$$= 17{,}296$$

Using the Counting Principle:

$6 \times 17{,}296 = 103{,}776$ ways to do both

- **Solution (cont.):**

2. Probability of choosing exactly 2 kings:

$$= \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{103{,}776}{2{,}598{,}960} = 0.040 = 4.0\%$$

3. There are 13 hearts, and the number of ways to choose 5 of them is the number of combinations of 13 items taken 5 at a time:

$$\text{Combinations of 13 items taken 5 at a time} = \frac{13!}{5!\,(13-5)!} = 1287$$

Probability of choosing all five hearts:

$$= \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{1287}{2{,}598{,}960} = 0.0005 = 0.05\%$$