# Learning Objectives:

- Know how to use graphs as models and how to determine efficient paths.
  - Modeling with graphs
  - Euler circuits
  - Degrees of vertices and Euler's Theorem

# 7.1 Modeling with graphs and finding Euler circuits.

- **Graph properties**

1. An edge cannot start and end at the same vertex. (See Figure 7.2)

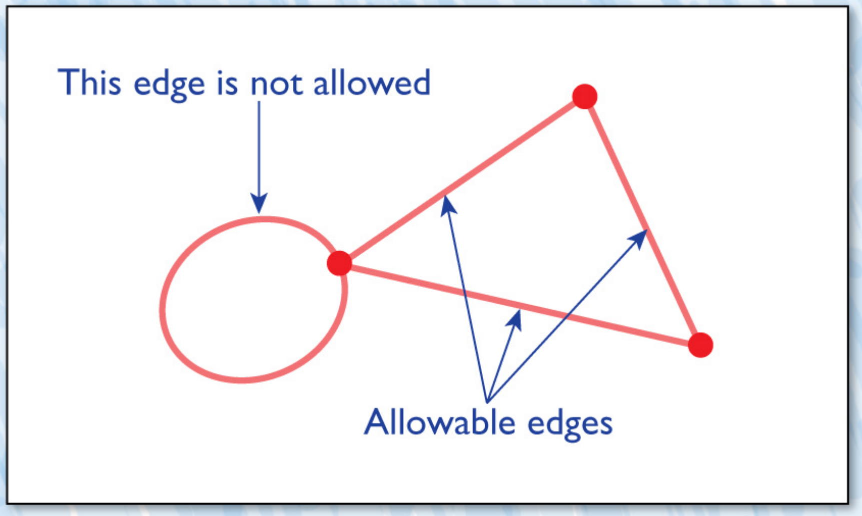


**FIGURE 7.2** An edge is not allowed to start and end at the same vertex.

- **Graph properties (cont.)**

2. The graphs are *connected* if each pair of vertices can be joined by a sequential collection of edges, a *path*.
3. The graph in Figure 7.3 is connected.
4. The one in Figure 7.4 is not connected.

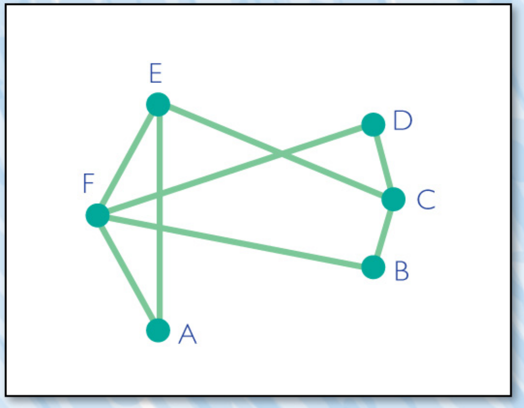


FIGURE 7.3 A connected graph.

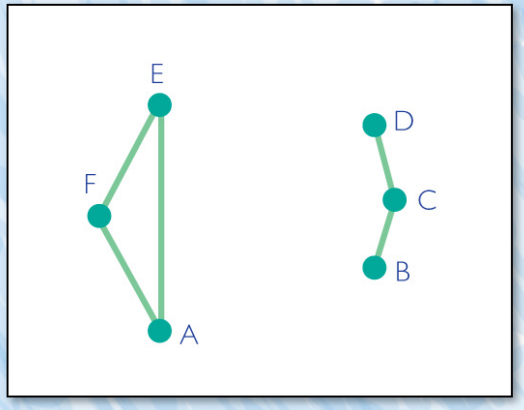


FIGURE 7.4 A disconnected graph.

7.1 Modeling with graphs and finding Euler circuits.

- **Graph properties (cont.)**

5. Graphs can be used to represent many situations. (See Figures 7.5 and 7.6)

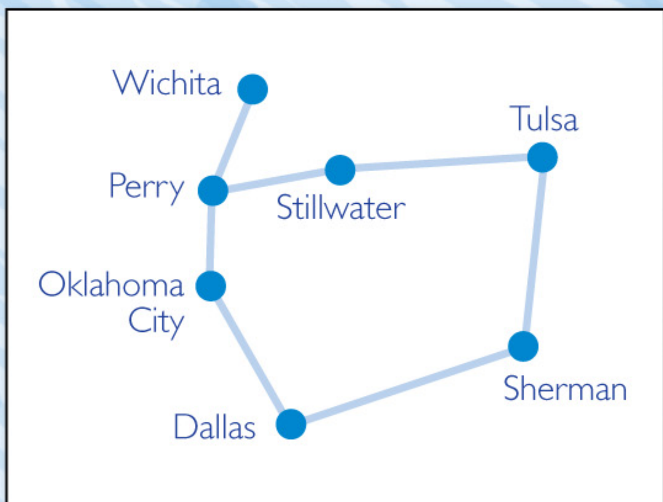


**FIGURE 7.5** A graph representing towns and roads.

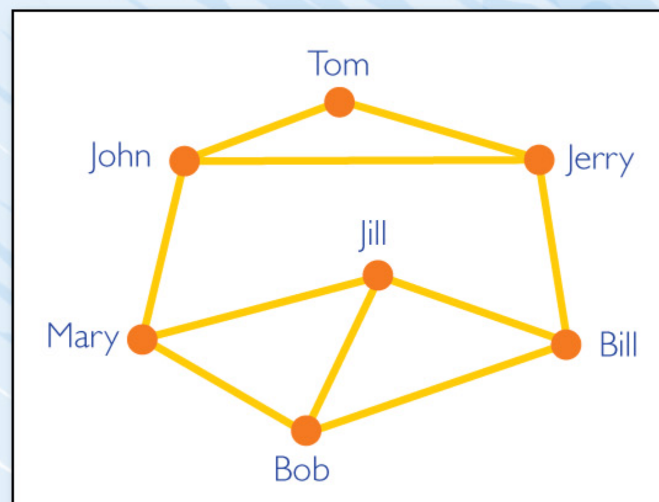


**FIGURE 7.6** Facebook friends.

# 7.1 Modeling with graphs and finding Euler circuits.

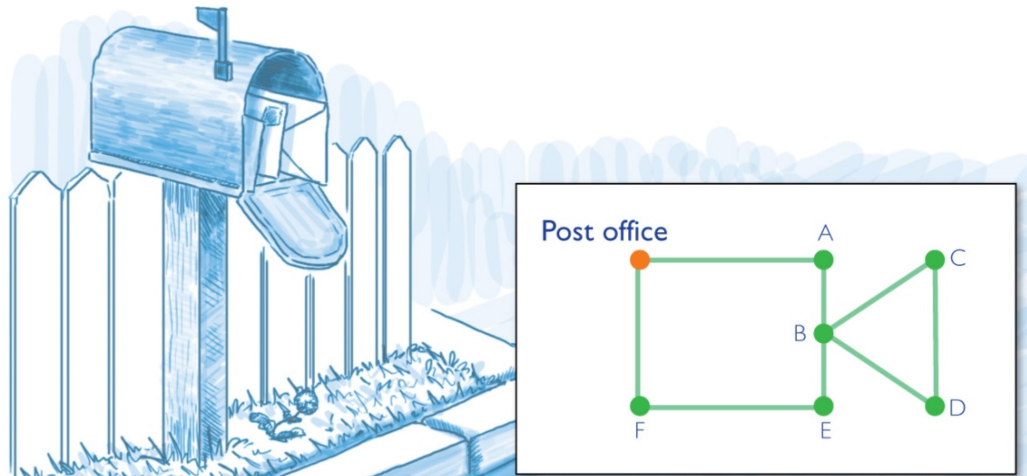


**FIGURE 7.8** A post office, streets, and intersections.

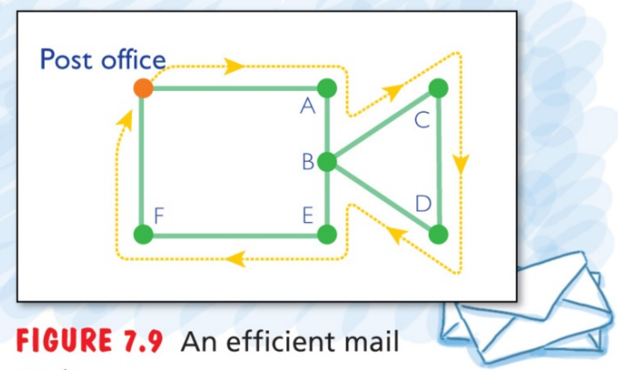


**FIGURE 7.9** An efficient mail route.

- A **circuit** or **cycle** in a graph is a path that begins and ends at the same vertex. An **Euler circuit** of **Euler cycle** is a circuit that traverses each edge of the graph exactly once.

# 7.1 Modeling with graphs and finding Euler circuits.

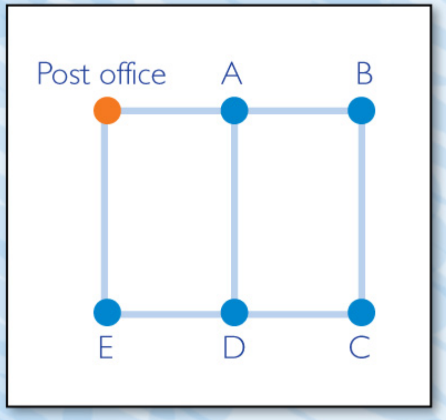


**FIGURE 7.10** No Euler circuit.

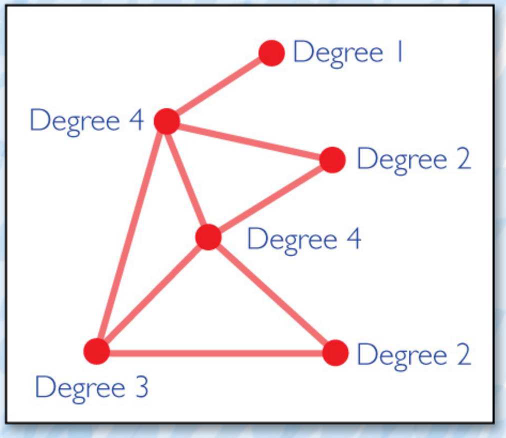


**FIGURE 7.11** The degrees of vertices.

- The **degree** of a vertex is the number of edges that touch that vertex. Some texts use **valence** instead of degree.
- The numbers in Figure 7.11 indicate the degree of each vertex.

# 7.1 Modeling with graphs and finding Euler circuits.

- **Example:** The graph in Figure 7.12 shows a simplified air route map. It shows connections that are available between various cities. In the context of this graph, what is the meaning of the degree of a vertex? Find the degree of each of the vertices.


**FIGURE 7.12** A simplified air route map.

## 7.1 Modeling with graphs and finding Euler circuits.

- **Solution:** Because each edge indicates a direct connection between cities, the degree of each vertex is the number of direct flights available from the given city. The degrees of the vertices are:
  - Atlanta: degree 4
  - Chicago: degree 2
  - Los Angeles: degree 2
  - Miami: degree 2
  - New York: degree 4

## 7.1 Modeling with graphs and finding Euler circuits.

- **Example:** Does the air route map in Figure 7.12 allow for a scenic trip that starts and ends at New York and flies along each route exactly once? Is there an Euler circuit for the graph in Figure 7.12? If an Euler circuit exists, use trial and error to find one.

- **Solution:** Referring back to the previous example, we see that the degree of each vertex is even. So Euler's Theorem guarantees the existence of an Euler circuit. Proceeding by trial and error, we find one such route:

  New York – Miami – Atlanta—New York–Chicago–Atlanta—Los Angeles—New York

  This route is shown in Figure 7.15.

  We note that this is not the only correct answer.

## 7.1 Modeling with graphs and finding Euler circuits.

- **Example:** Königsberg is an old city that is now part of Russia and has been renamed Kaliningrad. It covers both banks of the Pregel River as well as two islands in the river.

  The banks and islands were connected by seven bridges, as shown in Figure 7.16.

  Tradition has it that the people of Königsberg enjoyed walking, and they wanted to figure out how to start at home, go for a walk that crossed each bridge exactly once, and arrive back home.

  Nobody could figure how to do it. It was Leonhard Euler's solution of this problem that began the modern theory of graphs.

1. Make a graph that models Königsberg and its seven bridges.
2. Determine whether the desired walking path is possible.

# 7.1 Modeling with graphs and finding Euler circuits.

- **Example (cont.):**

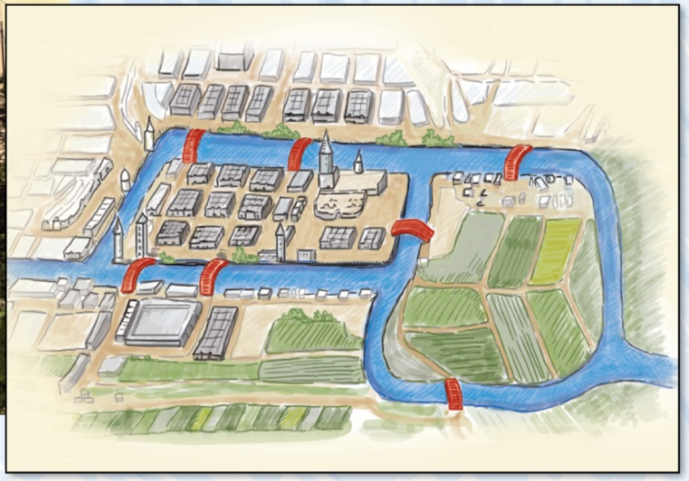

**FIGURE 7.16** Left: Königsberg (modern-day Kaliningrad).
Right: The bridges of Königsberg.

- **Solution:**

1. We place a vertex on each bank of the river and on both islands. We connect these vertices by paths across the seven bridges. (see Figure 7.17) We then obtain the graph shown in Figure 7.18 by showing only the vertices and edges we have drawn.
2. The desired walking path would be an Euler circuit for the graph in Figure 7.18. But because this graph has a vertex of odd degree, it has no Euler circuit.

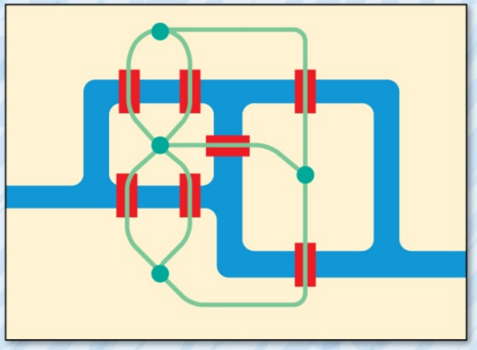

**FIGURE 7.17** Preparing a graph model.

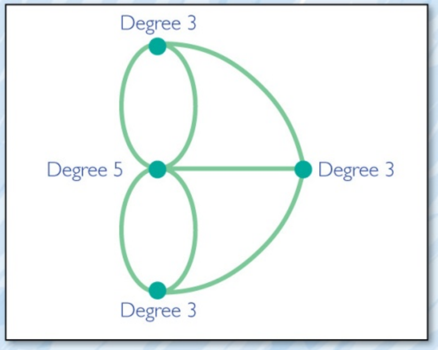


**FIGURE 7.18** A graphical representation of the seven bridges.

## Graphs and Euler circuits

1. A graph is a collection of vertices, some (or all) of which are joined by edges. Generally, we consider only connected graphs, and we do not allow edges to start and end at the same vertex.
2. A circuit in a graph is a path (a sequential collection of edges) that begins and ends at the same vertex. An Euler circuit is a circuit that uses each edge exactly once.
3. The degree of a vertex is the number of edges touching it.
4. A connected graph has an Euler circuit precisely when each vertex has even degree.

# Learning Objectives:

- Know how to determine circuits that traverse the vertices efficiently.
  - Making Hamilton circuits
  - The traveling salesman problem
  - The complexity of the traveling salesman problem
  - Nearest-neighbor algorithm
  - The cheapest link algorithm

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- A **Hamilton circuit** in a graph is a circuit that visits each vertex exactly once.

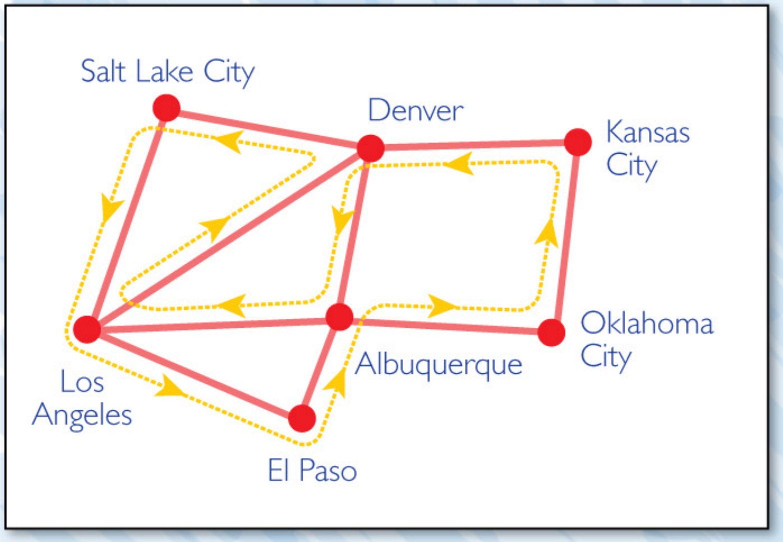


**FIGURE 7.61** An Euler circuit.

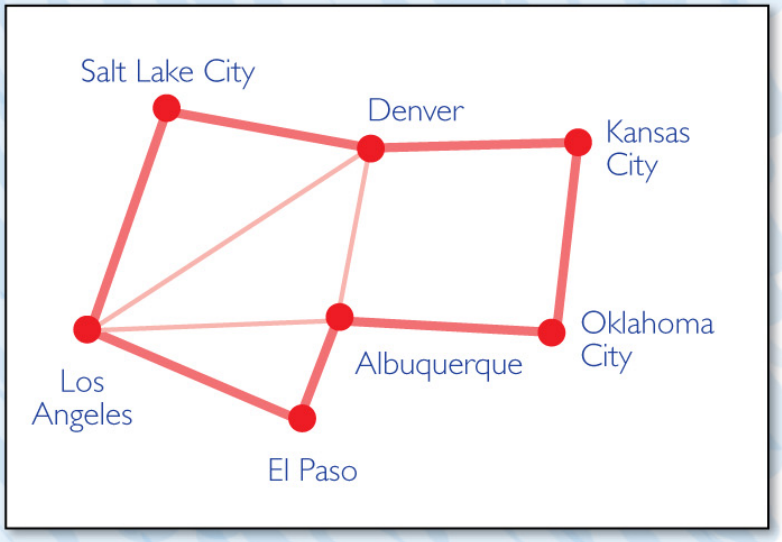


**FIGURE 7.62** A Hamilton circuit shown in heavier edges.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

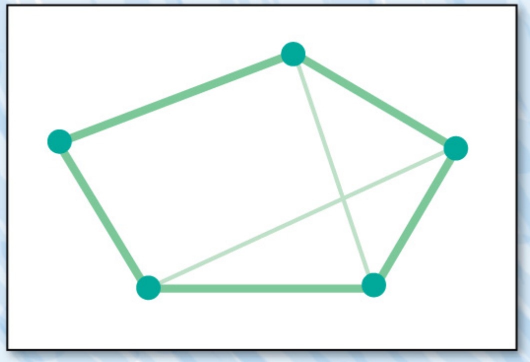

**FIGURE 7.63** One Hamilton circuit.

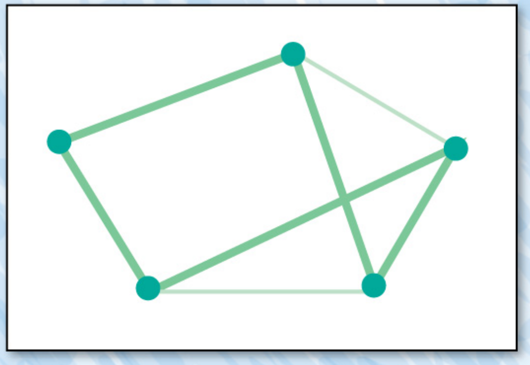

**FIGURE 7.64** Another Hamilton circuit.

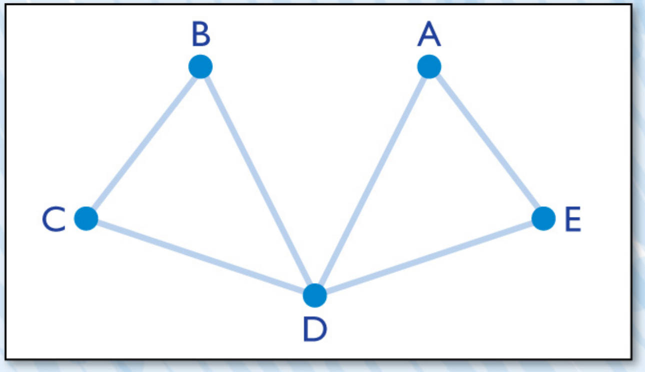


**FIGURE 7.65** A graph with no Hamilton circuit.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

**Vertices of Degree 2 and Hamilton Circuits**

If a graph has a vertex of degree 2, then each edge meeting that vertex must be part of any Hamilton circuit.

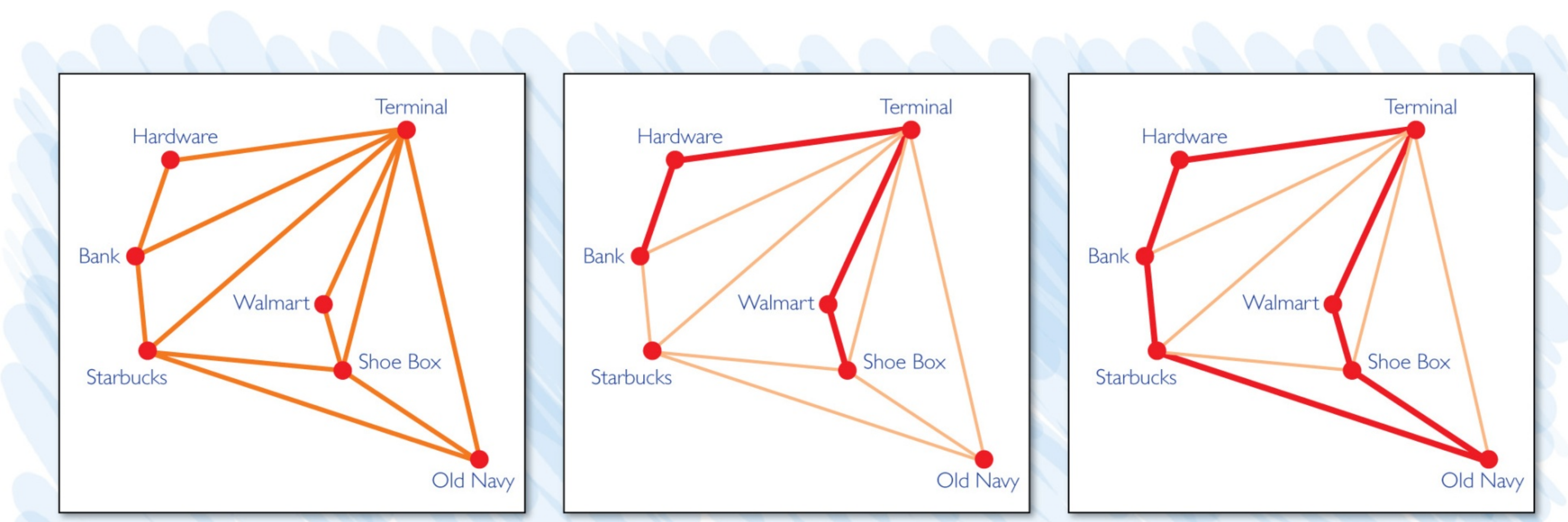


**FIGURE 7.66** A delivery route problem.

**FIGURE 7.67** We must use edges that meet vertices of degree 2.

**FIGURE 7.68** Completing the delivery route.

## Chapter 7 Graph Theory

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Example:** Figure 7.69 shows hiking trails and points of interest in Yellowstone National Park. We want to find a path that starts and ends at Shoshone Lake and visits each point of interest exactly once. (We want to find a Hamilton circuit.) Either find a Hamilton circuit or explain why no such circuit exists.

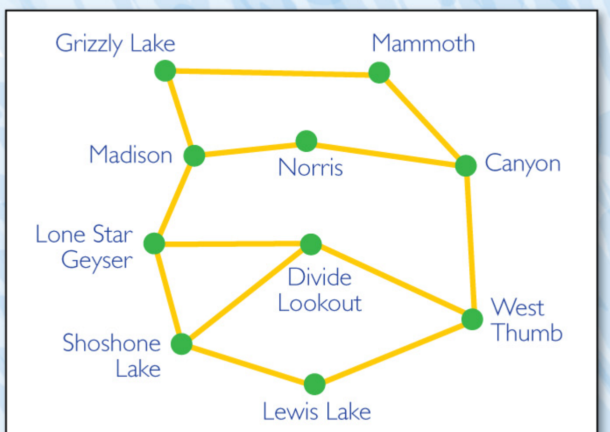


**FIGURE 7.69** Hiking paths.

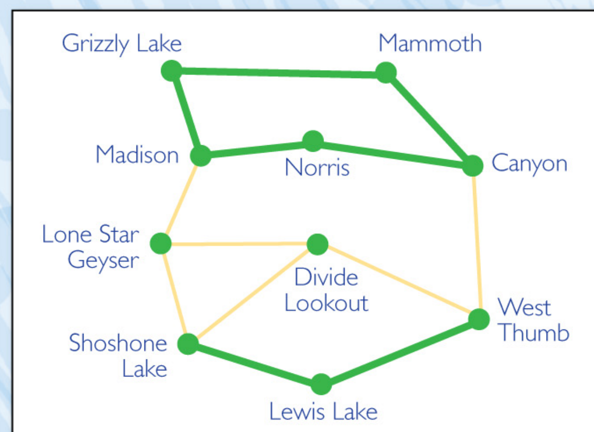


**FIGURE 7.70** Edges meeting vertices of degree 2 must be used.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:** The vertices Grizzly Lake, Mammoth, Norris, and Lewis all have degree 2. Thus in any Hamilton circuit, we must use the hiking paths that touch these vertices. Such a circuit cannot be part of any larger Hamilton circuit, and we conclude that it is not possible to take the desired walking path.

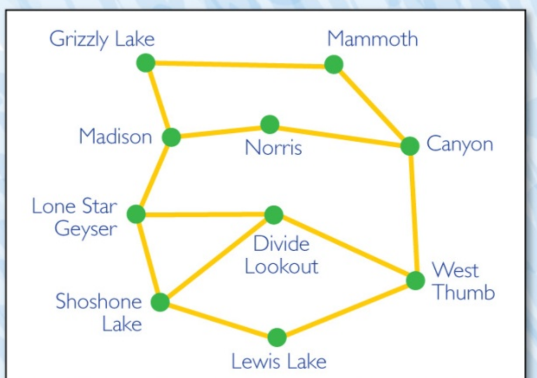


FIGURE 7.69 Hiking paths.

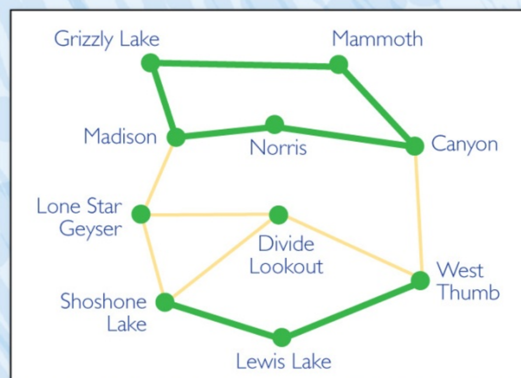


FIGURE 7.70 Edges meeting vertices of degree 2 must be used.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

## Hamilton Circuits

1. A Hamilton circuit is a circuit that visits each vertex exactly once.
2. There is no set procedure for determining whether Hamilton circuits exist.
   Here is one helpful observation: If a graph has a vertex of degree 2, then each edge meeting that vertex must be part of any Hamilton circuit.
3. A Hamilton circuit cannot contain a smaller circuit.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Example:** The graph in Figure 7.73 shows a delivery map for a trucking firm based in Kansas City. The firm needs a shortest route that will start and end in Kansas City and make stops in Houston, Phoenix, and Portland. That is, the trucking firm needs a solution of the traveling salesman problem for this map. Calculate the mileage for each possible route to find the solution.

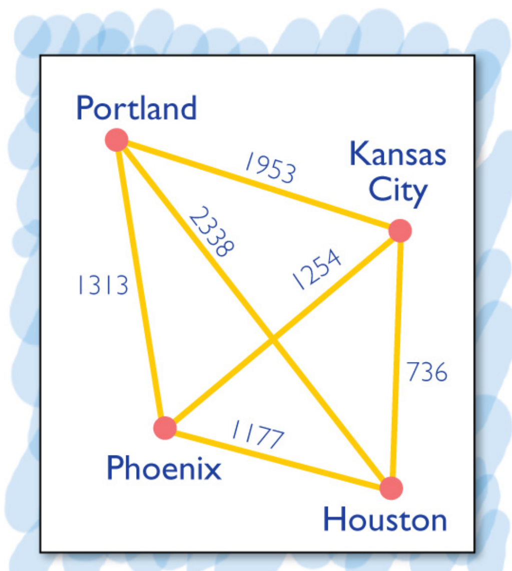


**FIGURE 7.73** Delivery map.

## 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:** There are six possible routes altogether, but we need to list only three because we gain no new information by looking at the reverse of a route:

- Kansas City–Houston–Phoenix–Portland–Kansas City: Distance 5179 miles
- Kansas City–Phoenix–Houston–Portland–Kansas City: Distance 6722 miles
- Kansas City–Phoenix–Portland–Houston–Kansas City: Distance 5641miles

The shortest route of 5179 miles is the first one listed.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- In a **complete graph**, each vertex is connected to every other vertex by an edge. The **traveling salesman problem** applies to complete graphs for which a distance is assigned to each edge. The problem is to find a shortest Hamilton circuit.

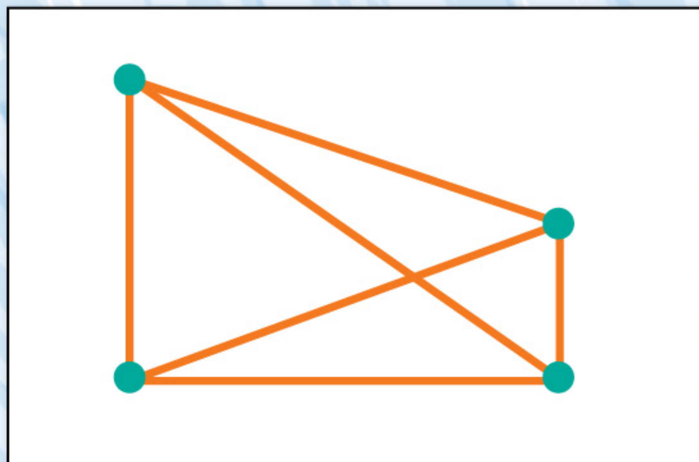

**FIGURE 7.75** A complete graph on four vertices.

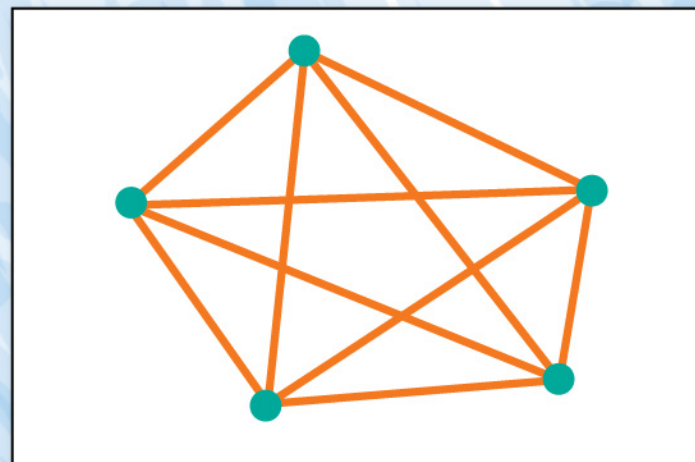

**FIGURE 7.76** A complete graph on five vertices.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- The **nearest-neighbor algorithm** constructs a Hamilton circuit in a complete graph by starting at a vertex. At each step, it travels to the nearest vertex not already visited (except at the final step, where it returns to the starting point). If there are two or more vertices equally nearby, any one of them may be selected.

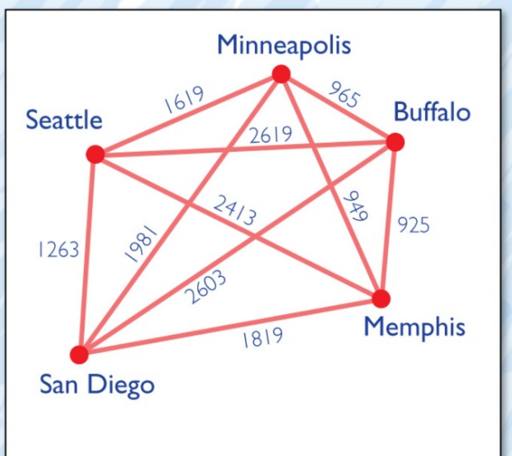


**FIGURE 7.78** Traveling salesman problem for five cities.

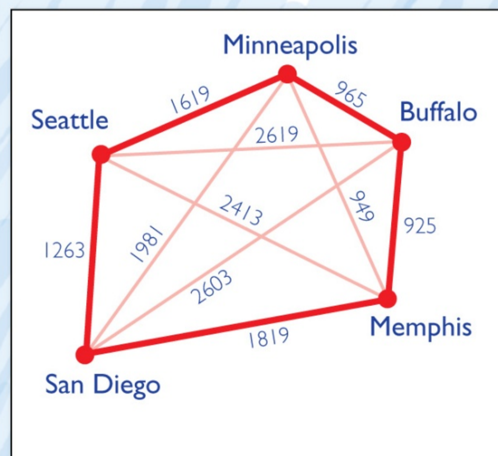


**FIGURE 7.79** Result of the nearest-neighbor algorithm.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Example:** The local school board needs to visit the high school, middle school, primary school, and kindergarten each day. A map illustrating the office and schools, along with distances in miles, is shown in Figure 7.80.

1. Find a shortest path by listing all possible routes starting and ending at the office.
2. Use the nearest-neighbor algorithm starting at the office to approximate a shortest Hamilton Circuit.

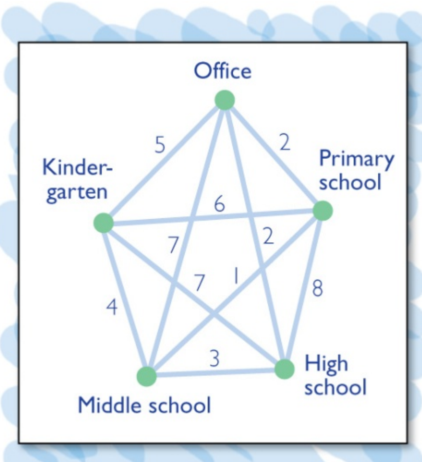


**FIGURE 7.80** A map of schools.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:** There are 12 possible routes if we do not include reverse routes.

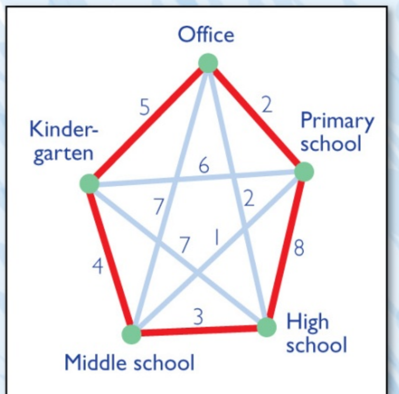


**FIGURE 7.81** 22 miles.

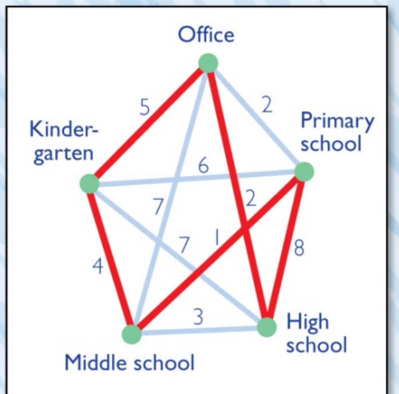


**FIGURE 7.82** 20 miles.

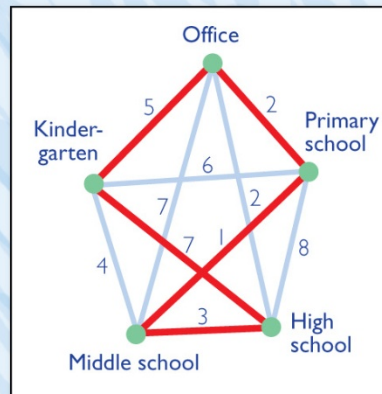


**FIGURE 7.83** 18 miles. (This is one of two routes that are found by the nearest-neighbor algorithm in part b.)

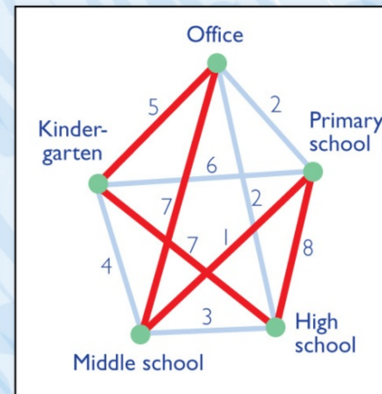


**FIGURE 7.84** 28 miles.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:** There are 12 possible routes if we do not include reverse routes.

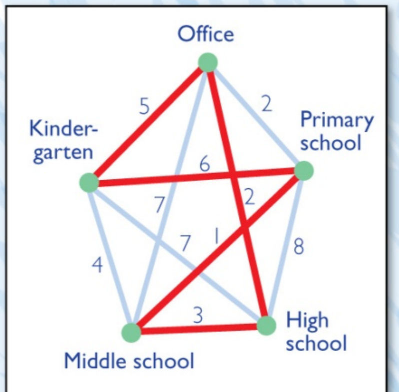


**FIGURE 7.85** 17 miles. (This is one of two routes that are found by the nearest-neighbor algorithm in part b.)

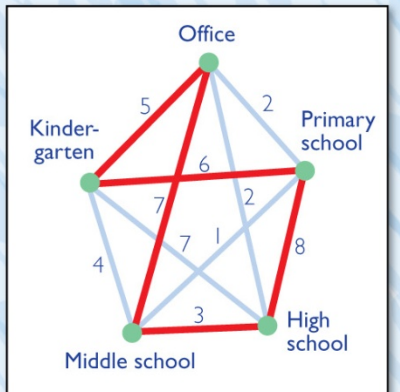


**FIGURE 7.86** 29 miles.

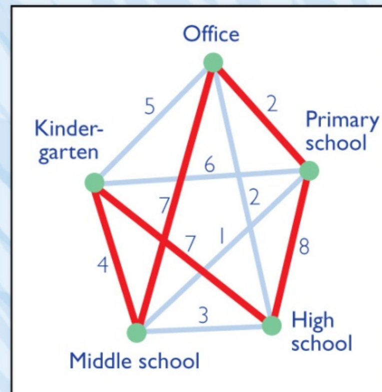


**FIGURE 7.87** 28 miles.

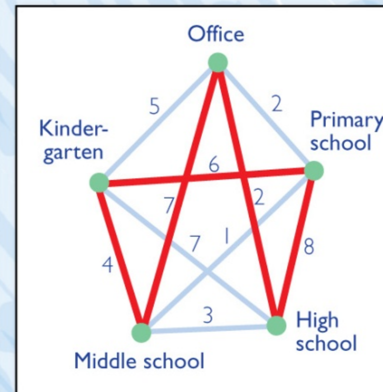


**FIGURE 7.88** 27 miles.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:** There are 12 possible routes if we do not include reverse routes.

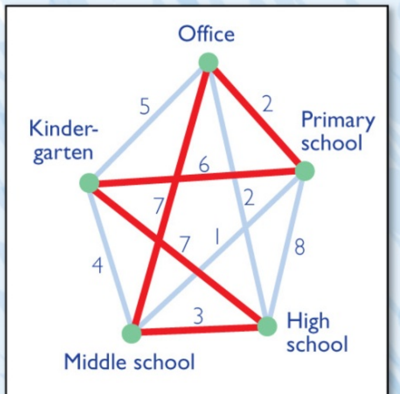

FIGURE 7.89 25 miles.

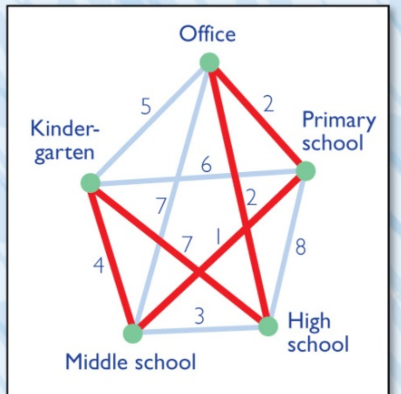

FIGURE 7.90 16 miles.

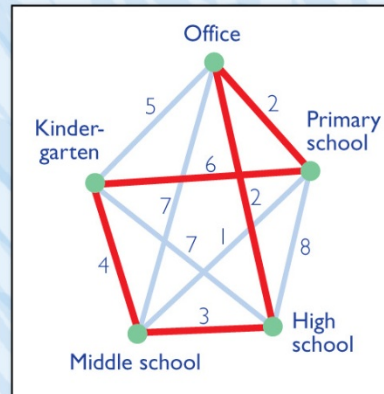

FIGURE 7.91 17 miles.

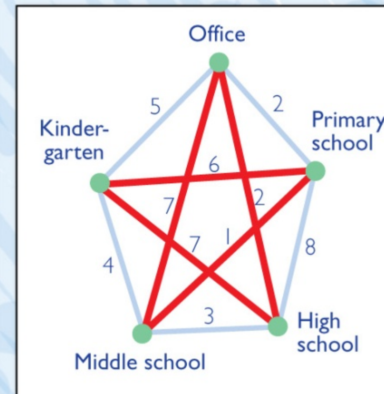

FIGURE 7.92 23 miles.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- Another method used to find approximate solutions of the traveling salesman problem is the **cheapest link algorithm**.

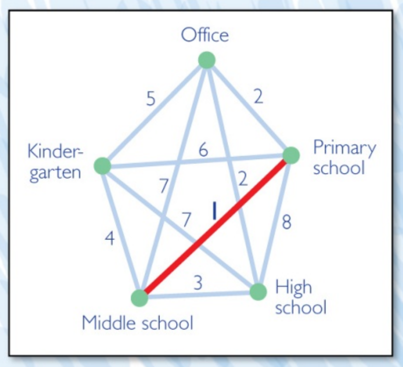


**FIGURE 7.93** Selecting the shortest edge in the graph.

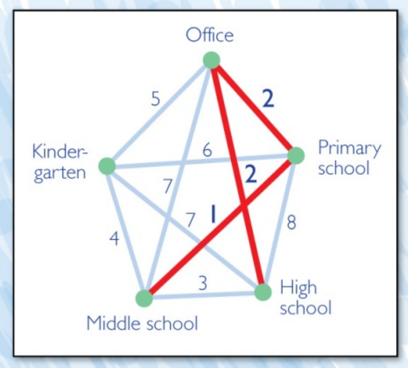


**FIGURE 7.94** Adding two more edges.

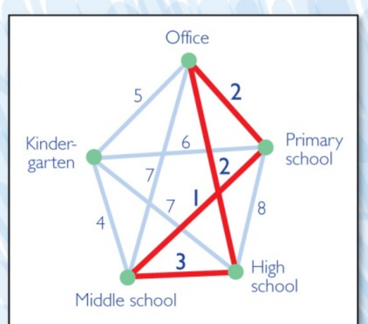


**FIGURE 7.95** Choosing the 3-mile edge results in a circuit.

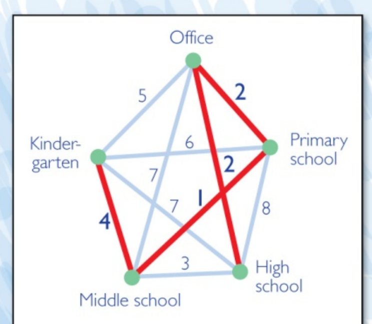


**FIGURE 7.96** Use the 4-mile edge instead.

- **Example:** We run a trucking company that makes deliveries in Seattle, Minneapolis, Buffalo, Memphis, and San Diego. The map is shown in Figure 7.99. Use the cheapest link algorithm to find an approximate solution of the traveling salesman problem.

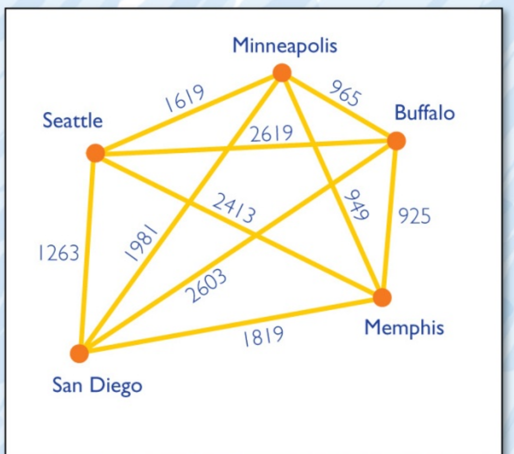


**FIGURE 7.99** Five cities.

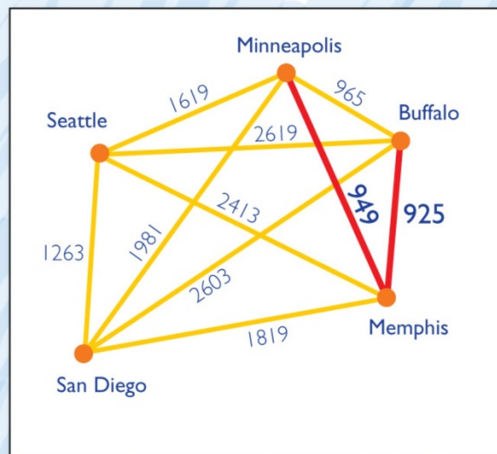


**FIGURE 7.100** Choosing the first two segments of the route.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

- **Solution:**

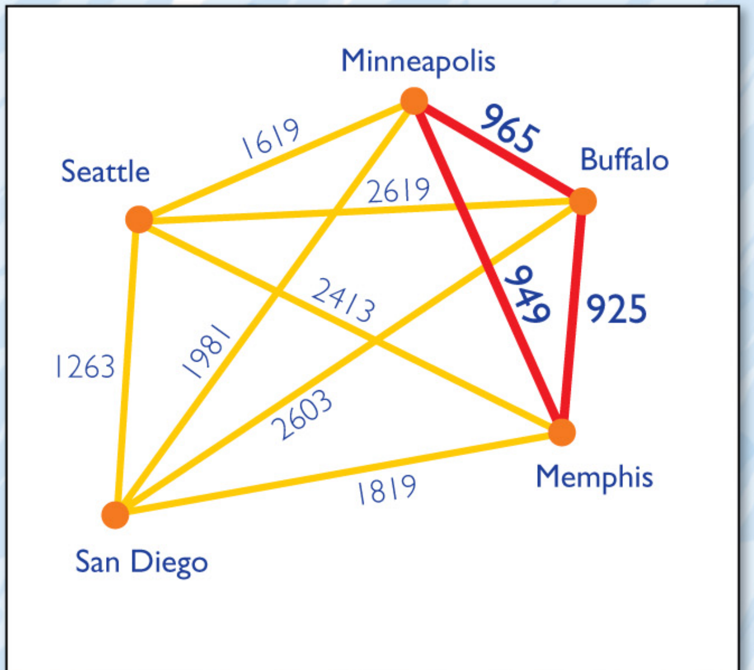


**FIGURE 7.101** Adding the edge from Minneapolis to Buffalo makes a circuit.

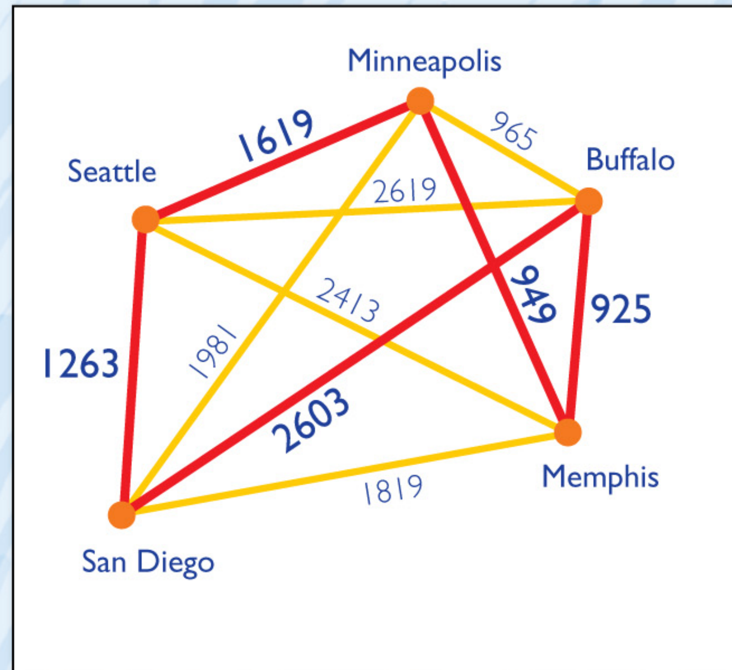


**FIGURE 7.102** The completed route.

# 7.2 Hamilton circuits and traveling salesmen: Efficient routes.

**The Traveling Salesman Problem**

1. A complete graph is one in which each pair of vertices is joined by an edge.
2. If a distance (more generally, a value) is assigned to each edge, the traveling salesman problem is to find a shortest Hamilton circuit.
3. Solving the traveling salesman problem is difficult when there are more than just a few vertices.
   However, there are algorithms that can give an approximate solution.
   One such is the nearest-neighbor algorithm.
   Another is the cheapest link algorithm.

# Chapter 7 Graph Theory: **Chapter Summary**

- **Modeling with graphs and finding Euler circuits**
  - Modeling with graphs
  - Euler circuits
  - Degrees of vertices and Euler's Theorem
- **Hamilton circuits and traveling salesmen: Efficient routes**
  - Making Hamilton circuits
  - Complete graphs: the traveling salesman problem
  - The *nearest-neighbor algorithm*, the *cheapest link algorithm*
- **Trees**