

# Quantitative Literacy: Thinking Between the Lines

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## Chapter 9: Geometry

# Chapter 9 Geometry

## Lesson Plan

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- ▶ Perimeter, area, and volume: How do I measure?
- ▶ Proportionality : Changing the scale
- ▶ Symmetries : Form and patterns

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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## Learning Objectives:

- ▶ Calculate perimeters, areas, volumes, and surface areas of familiar figures.
  - ▶ Finding the area: Reminders about circles, rectangles, and triangles
  - ▶ Applications of basic geometry formulas
  - ▶ Three-dimensional objects

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

- ▶ **Circle:** a figure where all points are a fixed distance, the *radius*, from a fixed point, the *center*.

- ▶ For a circle of radius  $r$ :

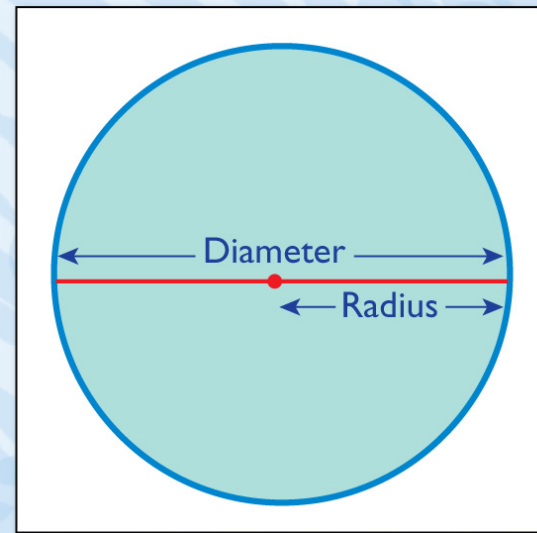
$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

The number  $\pi \approx 3.14159$ .

The formula for the circumference of a circle can be rewritten as:

$$\frac{\text{Circumference}}{\text{Diameter}} = \pi$$



**FIGURE 9.2** Radius and diameter of a circle.

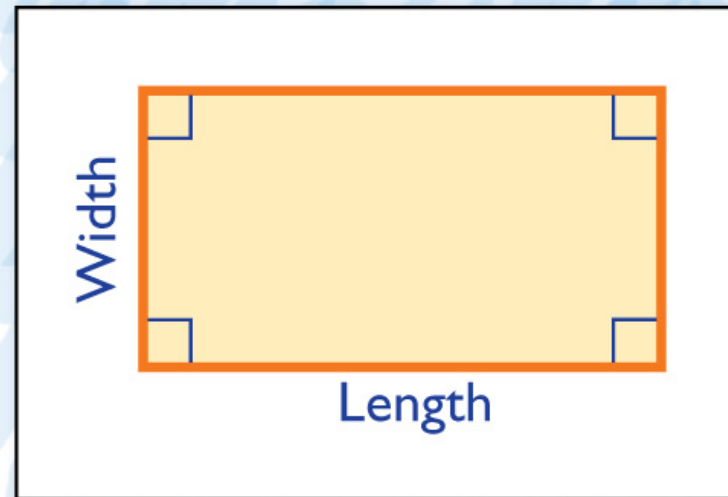
This tells us that the ratio of circumference to diameter is always the same no matter which circle you study. The ratio is the same for the equator of earth and the equator of a baseball.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

- ▶ **Rectangles:** A rectangle has four right angles, and opposite sides are equal. Here are two basic formulas:

$$\text{Area} = \text{Length} \times \text{Width} \quad \text{Perimeter} = 2 \times \text{Length} + 2 \times \text{Width}$$



**FIGURE 9.3** Opposite sides of a rectangle are of equal length.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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- ▶ **Triangles:** A triangle is a figure with three sides.

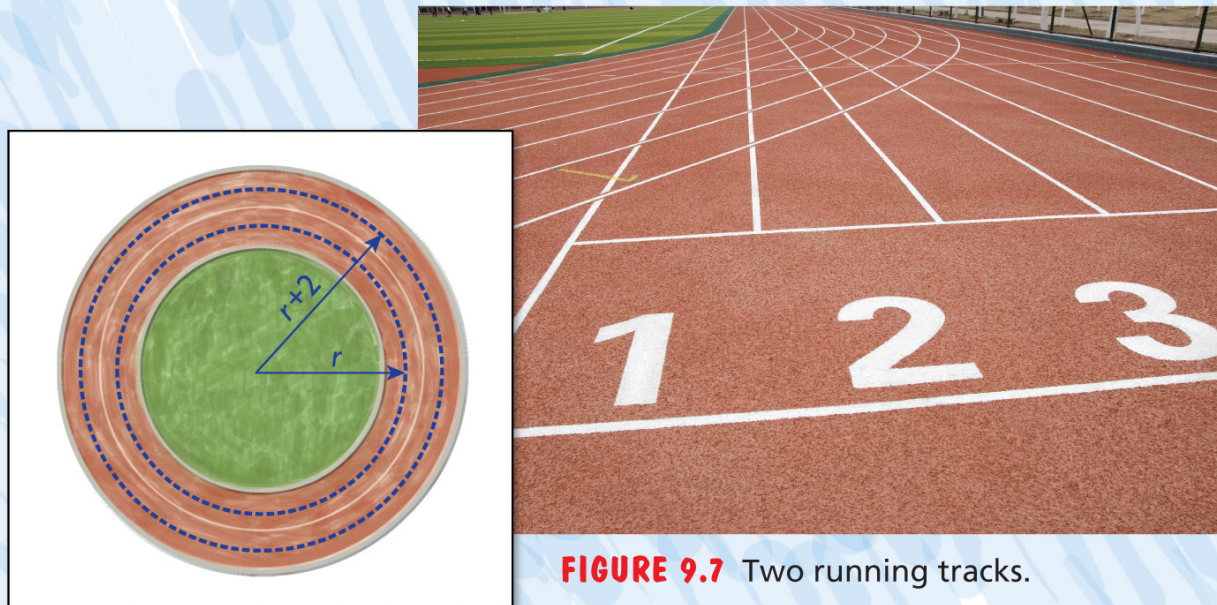
To find the area of a triangle, we select any one of the three sides and label it the *base*. Then we find the *height* by starting at the vertex opposite the base and drawing a line segment that meets the base in a right angle.

- ▶ The **perimeter** of a geometric figure is the distance around it.
  1. A circle, the perimeter is **circumference** and equals  $2\pi$  times the radius.
  2. A rectangle, the perimeter is the sum of the lengths of its four sides.
  3. A triangle, the perimeter is the sum of the lengths of its three sides.
- ▶ The **area** of a geometric figure measures the region enclosed by the figure.
  1. A circle, the area is  $\pi$  times the radius squared.
  2. A rectangle, the area is the product of the length and the width.
  3. A triangle, the area is one-half the product of the length of the base times the length of the height.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

- ▶ **Example:** Suppose two runners A and B are side-by-side, 2 feet apart, on a circular track, as seen in Figure 9.7. If they both run one lap, staying in their lanes, how much farther did the outside runner B go than the inside runner A? ( Note that no information was given about the diameter of the track.)



**FIGURE 9.7** Two running tracks.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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► **Solution:**

Let  $r$  denote the radius of the inside lane, where A is running.

Then  $r + 2$  is the radius of the outside lane, where B is running.

The distance covered by runner A:

$$\text{Length of inside track} = 2\pi r$$

The distance covered by B:

$$\begin{aligned}\text{Length of outside track} &= 2\pi(r + 2) \\ &= 2\pi r + (2\pi \times 2) = \text{Length of inside track} + 4\pi\end{aligned}$$

Therefore, the distances covered by the two runners differ by  $4\pi$  or about 12.6 feet. It may appear that a longer inside track would cause a greater difference in the distance the runners travel.

But the difference is the same, 12.6 feet, whether the inside track is 100 yards in diameter or 100 miles in diameter.



## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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- ▶ **Example:** At a local Italian restaurant a 16-inch-diameter pizza costs \$15 and a 12-inch-diameter pizza costs \$10.

Which one is the better value?

- ▶ **Solution:** The area of each pizza in terms of its radius  $r$ :

$$\text{Area} = \pi r^2$$

The 16-inch-diameter pizza:  $r = 8$  inches

$$\text{Area of 16-in. -diameter pizza} = \pi \times (8 \text{ in.})^2 = 64\pi \text{ in.}^2$$

The 12-inch-diameter pizza:  $r = 6$  inches

$$\text{Area of 12-in. -diameter pizza} = \pi \times (6 \text{ in.})^2 = 36\pi \text{ in.}^2$$

The larger pizza costs:  $\$15/64\pi$  square inches or  $\$0.075$  per  $\text{in.}^2$ .

The smaller pizza costs:  $\$10/36\pi$  square inches or  $0.088$  per  $\text{in.}^2$ .

Thus, the larger pizza is the better value.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

- ▶ Labeling the dimensions of a box as *length*, *width*, and *height* as shown in Figure 9.14:

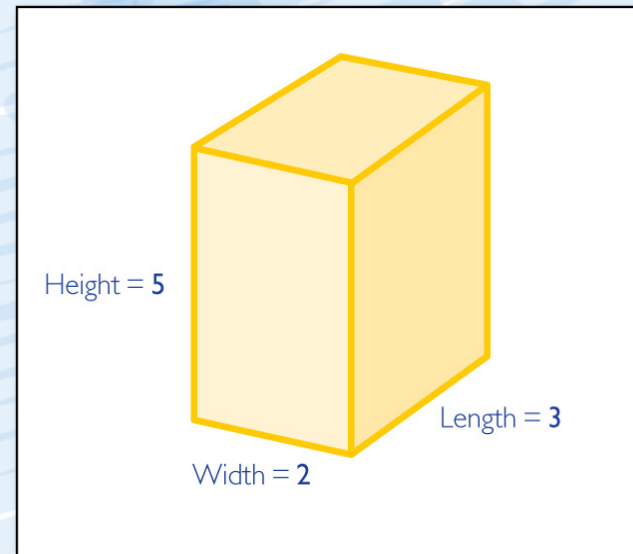
$$\text{Volume of a box} = \text{Length} \times \text{Width} \times \text{Height}$$

For example, the volume of the box in Figure 9.14 is:

$$2 \text{ units} \times 3 \text{ units} \times 5 \text{ units} = 30 \text{ cubic units}$$

Note that the volume of a box can also be expressed as the area of the base times the height:

$$\text{Volume} = \text{Area of base} \times \text{Height}$$



**FIGURE 9.14** Dimensions of a box.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

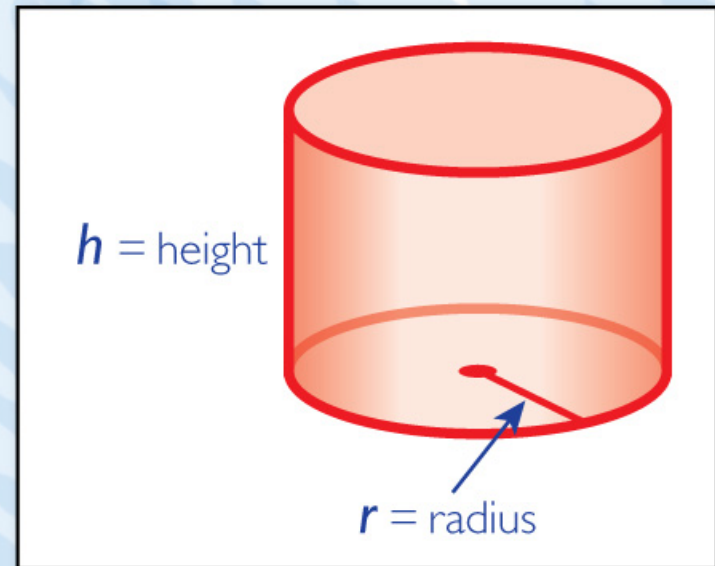
$$\text{Volume} = \text{Area of base} \times \text{Height}$$

This formula also holds for any three-dimensional object with uniform cross sections, such as a cylinder.

A cylinder has uniform circular cross sections, and a box has uniform rectangular cross sections.

For the cylinder shown in Figure 9.15, the radius is  $r$ , so the area of the base is  $\pi r^2$ . Using  $h$  for the height,

$$\begin{aligned}\text{Volume} &= \text{Area of base} \times \text{Height} \\ &= \pi r^2 h\end{aligned}$$



**FIGURE 9.15** A cylinder.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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▶ **Example:** What is the volume of a cylindrical wading pool that is 6 feet across and 15 inches high?

▶ **Solution:** Because the pool is 6 feet across, the diameter of the circular base is 6 feet. The diameter is given in feet and the height is given in inches, so we need to convert units for one of them.

Let's change the diameter of 6 feet to 72 inches.

Then the radius  $r = 36$  inches, and the height  $h = 15$  inches.

$$\begin{aligned}\text{Volume} &= \text{Area of base} \times \text{Height} \\ &= \pi r^2 h \\ &= \pi(36 \text{ inches})^2 \times 15 \text{ inches}\end{aligned}$$

This is about 61,073 cubic inches.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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- ▶ **Example:** Cake batter rises when baked. The best results for baking cakes occur when the batter fills the pan to no more than two-thirds of the height of the pan. Suppose we have 7 cups of batter, which is about 101 cubic inches. We have a pan that is 2 inches high and has a square bottom that is 9 inches by 9 inches. Is this pan large enough?



## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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- ▶ **Solution:** This pan forms the shape of a box, and we want to find the volume based on two-thirds of the full height 2 inches. Thus we use a height of:

$$\text{Height} = \frac{2}{3} \times 2 = \frac{4}{3} \text{ inches}$$

We find:

$$\begin{aligned} \text{Volume} &= \text{Length} \times \text{Width} \times \text{Height} \\ &= 9 \text{ inches} \times 9 \text{ inches} \times \frac{4}{3} \text{ inches} \\ &= 108 \text{ cubic inches} \end{aligned}$$

This pan will easily hold batter with a volume of 101 cubic inches.

## Chapter 9 Geometry

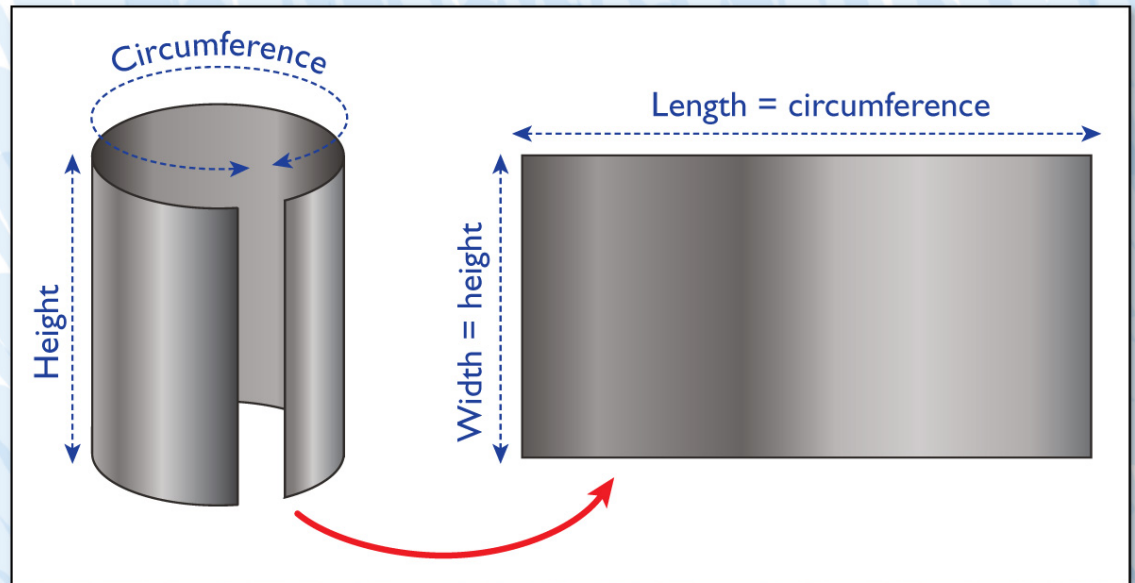
### 9.1 Perimeter, area, and volume: How do I measure?

- ▶ To find the **surface area** of a cylinder of radius  $r$  and height  $h$  (excluding the top and bottom), think of the cylinder as a can that we split lengthwise and roll out flat. (See Figure 9.16)

This gives a rectangle with width  $h$  and length equal to the circumference of the circular base.

The circumference of the base is  $2\pi r$ , so the surface area of the cylinder (excluding the top and bottom):

Surface area of cylinder  
 $= 2\pi r h$



**FIGURE 9.16** Splitting the cylindrical side of a can and rolling it out flat.

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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▶ **Example:** A tin can has a radius of 1 inch and a height of 6 inches.

1. How much liquid will the can hold?
2. How much metal is needed to make the can?

▶ **Solution:**

1. The base is a circle of radius 1:

$$\text{Area of base} = \pi \times 1^2 = \pi \text{ square inches}$$

The height is 6 inches, so the volume is:

$$\text{Volume} = \text{Area of base} \times \text{Height} = \pi \times 6 \text{ cubic inches}$$

This is about 18.8 cubic inches.

2. The metal needed to make the can consists of the top, bottom, and cylindrical side.



## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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► **Solution (cont.):**

2. We already found that the base has area  $\pi$  square inches.

Area of top and bottom =  $2\pi$  square inches

To find the area of the cylindrical side, we use the formula  $2\pi rh$  for the surface area of a cylinder:

Area of side =  $2\pi \times 1 \times 6 = 12\pi$  square inches

The total area includes the top and bottom of the can:

$$2\pi + 12\pi = 14\pi \approx 44.0 \text{ square inches}$$

## Chapter 9 Geometry

### 9.1 Perimeter, area, and volume: How do I measure?

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#### **Volumes and Surface Areas**

- The volume of a box is:

$$\text{Volume of a box} = \text{Length} \times \text{Width} \times \text{Height}$$

- The volume of a cylinder is:

$$\text{Area of base} \times \text{Height}$$

If the cylinder has radius  $r$  and height  $h$ , this equals  $\pi r^2 h$ .

- The surface area of a cylinder (excluding the top and bottom) of radius  $r$  and height  $h$  is  $2\pi r h$ .

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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#### Learning Objectives:

- ▶ Understand the concept of proportionality
- ▶ How a change in the dimensions of a figure affects its area or volume.
  - ▶ Properties of proportionality
  - ▶ The golden ratio  $\phi$
  - ▶ Golden rectangles in art and architecture
  - ▶ Similar triangles
  - ▶ Other types of proportionality



## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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- ▶ One variable quantity is **(directly) proportional** to another if it is always a fixed (nonzero) constant multiple of the other.
- ▶ The quantity  $A$  is proportional to the quantity  $B$  if there is a nonzero constant  $c$  such that

$$A = cB$$

where the number  $c$  is the **constant of proportionality**.

- ▶ **Example:** If you are driving at a constant speed of 60 miles per hour, then the distance you travel is proportional to the time you spend driving.

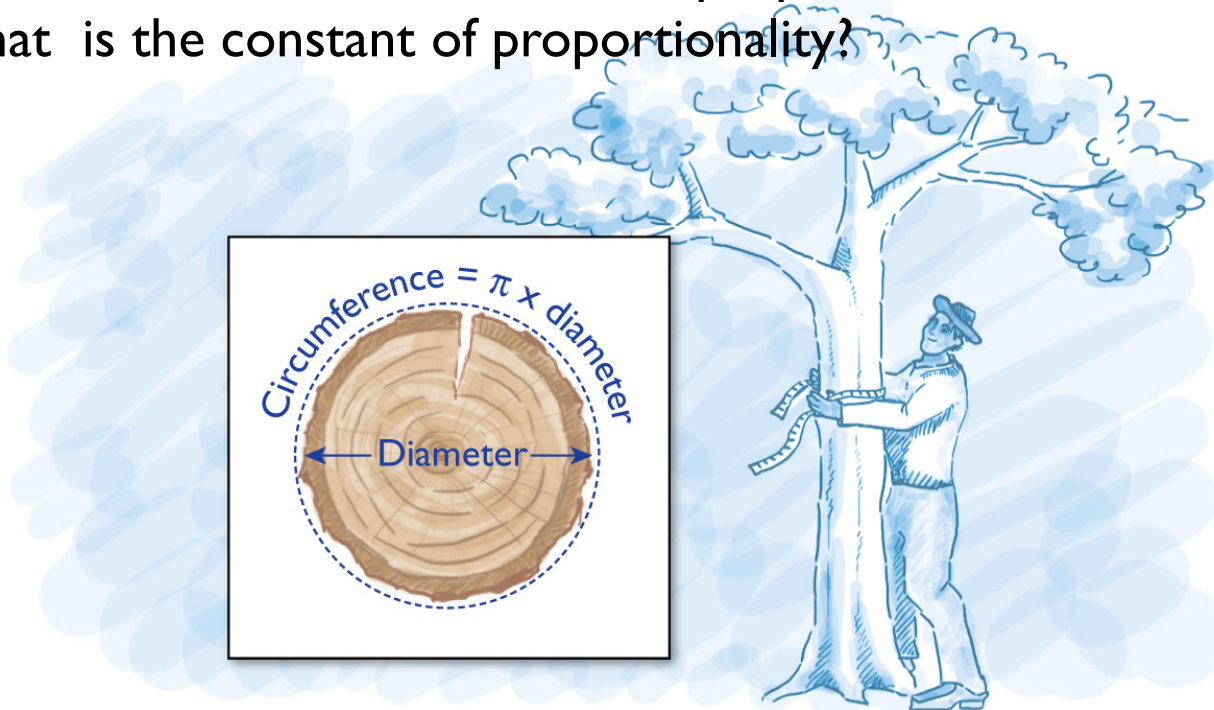
If the distance is measured in miles and time is in hours:

$$\text{Distance} = 60 \times \text{Time}$$

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

- ▶ **Example:** Biologists and foresters use the number of the trees and their diameters as one measure of the condition and age of a forest. Measuring the diameter of a tree directly is not easy; however, the circumference is easy to measure by running a tape measure around the tree. Is the diameter of a tree proportional to its circumference? What is the constant of proportionality?



**FIGURE 9.36** Foresters find the diameter of trees by measuring the circumference.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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- ▶ **Solution:** We know that circumference is proportional to diameter, and the relationship is

$$\text{Circumference} = \pi \times \text{Diameter}$$

We can rearrange this formula to find

$$\text{Diameter} = \frac{1}{\pi} \times \text{Circumference}$$

Thus, the diameter is proportional to the circumference, and the constant of proportionality is  $1/\pi$ .

**Note:** What happens to the circumference of a tree if the diameter is doubled?

$$\text{Circumference} = \pi \times \text{Diameter}$$

The result is:

$$2 \times \text{Circumference} = \pi \times (2 \times \text{Diameter})$$

Doubling the diameter causes the circumference to double as well.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

#### Properties of Proportionality

1. Two variable quantities are proportional when one is a (nonzero) constant multiple of the other. This multiple is the constant of proportionality. (Two quantities are proportional when their ratio is always the same.)
2. If a variable quantity  $A$  is proportional to another quantity  $B$  with proportionality constant  $c$ , then  $B$  is proportional to  $A$  with proportionality constant  $1/c$ .
3. If one of two proportional quantities is multiplied by a certain factor, the other one is also multiplied by that factor.  
(If  $y$  is proportional to  $x$ , then multiplying  $x$  by  $k$  has the result of multiplying  $y$  by  $k$ .)  
If one quantity doubles, then the other will double as well.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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- ▶ **Example:** It is a fact that the volume of a cylindrical can of diameter 3 inches is proportional to the height of the can. One such can is 4 inches high, and another is 12 inches high.

How do their volumes compare?

- ▶ **Solution:** If one of two proportional quantities is multiplied by a certain factor, the other one is also multiplied by that factor.

A change from 4 inches to 12 inches is a tripling of the height.

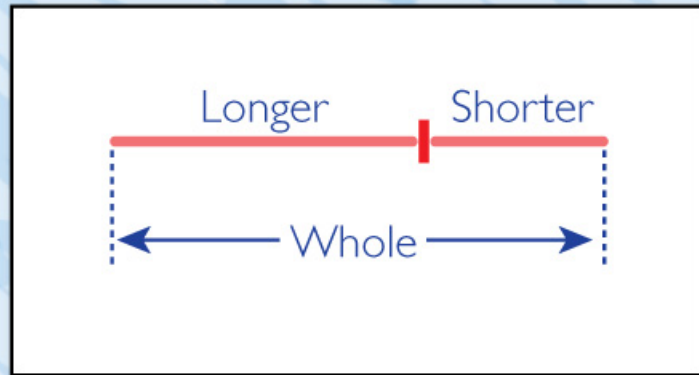
Therefore, the volume of the taller can is three times that of the shorter one.



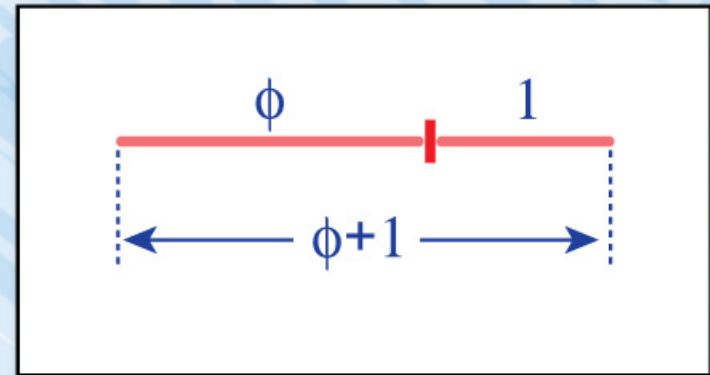
## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

- ▶ Cut the segment in Figure 9.37 so that the ratios  $\frac{\text{Longer}}{\text{Shorter}}$  and  $\frac{\text{Whole}}{\text{Longer}}$  are equal. This common ratio is the **golden ratio**.



**FIGURE 9.37** Segments whose ratio is the golden ratio  $\phi$ .



**FIGURE 9.38** Calculating  $\phi$ .

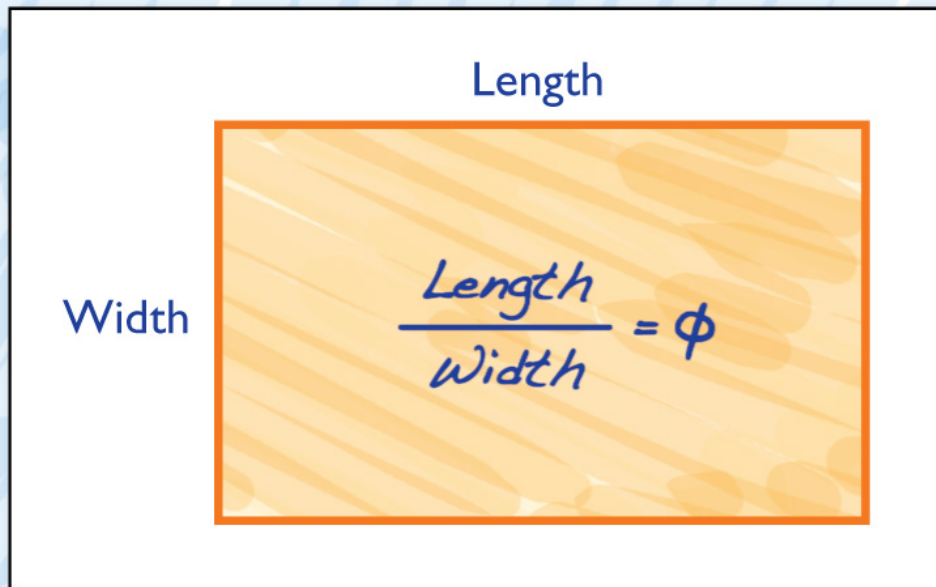
$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.62$$

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

- ▶ A **golden rectangle** is a rectangle for which the ratio of the length to the width is the golden ratio  $\phi$ .

That is, among golden rectangles the length is proportional to the width with constant of proportionality  $\phi$ . (See Figure 9.39)



**FIGURE 9.39** A golden rectangle:  $\frac{\text{Length}}{\text{Width}} = \phi$ .

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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- ▶ **Example:** The width (the shorter side) of a rectangle is 3 feet. What would the length have to be to make this a golden rectangle?

- ▶ **Solution:**

$$\frac{\text{Length}}{\text{Width}} = \phi,$$

we have:

$$\text{Length} = \text{Width} \times \phi = 3 \times 1.62 = 4.9 \text{ feet}$$

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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▶ **Example:** If the radius of a circle is doubled, what happens to the area?

▶ **Solution:**

If the radius is  $r$  and the area is  $A$ , the relationship is given by the formula:

$$A = \pi r^2$$

If the radius is doubled, we replace  $r$  by  $2r$ . This gives a new area of:

$$\pi(2r)^2 = 4\pi r^2 = 4A$$

The result is that the area is multiplied by 4.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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▶ **Example:** Suppose we have several boxes of different sizes, each of which is in the shape of a cube.

1. Is the volume of a box proportional to the length of one side of the box, the square of the length, or the cube of the length?

Find the constant of proportionality.

2. One of our boxes has sides that are twice as long as those of another box.

How much more does the larger box hold than the smaller box?

3. Is the surface area of a box proportional to the length of one side of the box, the square of the length, or the cube of the length?
4. The cost of wrapping paper for a box is proportional to the surface area. What happens to the cost of wrapping paper if the sides of the cube are doubled in length?

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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► **Solution:**

1. Let  $x$  denote the length of one side of the cube, and let  $V$  denote the volume. Because the box is cubical in shape, the length, width, and height are all the same,  $x$ :

$$V = \text{Length} \times \text{Width} \times \text{Height} = x \times x \times x = x^3$$

Thus, the volume is proportional to the cube of the length, and the constant of proportionality is 1.

2. As in part 1, we denote the length of a side by  $x$ . If the sides are doubled, we replace  $x$  by  $2x$ . This gives a new volume of

$$(2x)^3 = 2^3 x^3 = 8x^3$$

Thus, the larger volume is 8 times the smaller volume.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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▶ **Solution (cont.):**

3. The amount of paper needed is the surface area of the box. The surface of a cube has six square “faces.” If  $x$  represents the length of a side, the area of one face is  $x \times x = x^2$ .

The area of the surface of the cube is  $6x^2$ . This means that the surface area is proportional to the square of the length, with constant of proportionality 6.

4. We know that the cost is proportional to the surface area.

$$\text{New surface area} = 6 \times (2x)^2 = 4(6x^2)$$

The larger surface area is 4 times the other surface area.

Four times as much wrapping paper will be needed for the larger cube as for the smaller cube.

Therefore, the cost of wrapping paper is multiplied by 4 when the sides are doubled in length.

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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- ▶ **Example:** In a science fiction movie, an ape has grown to 100 times its usual height.
  1. The weight of an ape is taken to be proportional to the cube of its height. How does the weight of the overgrown ape compare to its original weight?
  2. The cross-sectional area of a limb is taken to be proportional to the square of the height. How does the cross-sectional area of a limb of the overgrown ape compare to the original area?
  3. The pressure on a limb is the weight divided by the cross-sectional area. Use parts 1 and 2 to determine how the pressure on a limb of the overgrown ape compares to the original pressure.



## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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► **Solution:**

- I. Let  $h$  denote the height of an ape and  $W$  the weight.

The weight is proportional to the cube of the height:

$$W = ch^3$$

where  $c$  is the constant of proportionality.

If the height is multiplied by 100, we replace  $h$  by  $100h$ :

$$\text{New weight} = c(100h)^3 = 100^3 \times ch^3$$

The original weight was  $ch^3$ .

Thus, the new weight is  $100^3$  times the original weight, the weight increases by a factor of  $100^3 = 1,000,000$  (one million).

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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► **Solution (cont.):**

2. Let  $A$  denote the cross-sectional area of a limb and let  $h$  denote the height. The cross-sectional area is proportional to the square of the height:

$$A = Kh^2$$

Here,  $K$  is the constant of proportionality.

If the height is multiplied by 100, we replace  $h$  by  $100h$ .

$$\text{New area} = K(100h)^2 = 100^2 \times Kh^2$$

The original area was  $Kh^2$ .

The new weight is  $100^2$  times the original area, the area increases by a factor of  $100^2 = 10,000$  ( ten thousand).

## Chapter 9 Geometry

### 9.2 Proportionality and similarity: Changing the scale

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► **Solution (cont.):**

3. Because the pressure on a limb is the weight divided by the cross-sectional area, the pressure will increase by a factor of

$$\frac{\text{Factor by which weight increases}}{\text{Factor by which area increases}} = \frac{100^3}{100^2} = 100$$

The pressure on a limb of the overgrown ape is 100 times the original pressure.

The tremendous increase in pressure on a limb means that the overgrown ape would collapse under its own weight.

Science fiction aside, such *scaling arguments* are used by biologists to study the significance of the size and shape of organisms.

# Chapter 9 Geometry: **Chapter Summary**

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- ▶ **Perimeter, area, and volume:** How do I measure?
  - ▶ Geometric objects can be measured in terms of perimeter (circumference), area and volume.
- ▶ **Proportionality and similarity:** Changing the scale
  - ▶ Proportional and its constant of proportionality
  - ▶ Golden rectangles
- ▶ **Symmetries and tiling:** Form and patterns
  - ▶ Rotational symmetry and reflectional symmetry about a line

