Dr. Katiraie Math 115A Notes section 3.2 (Solutions)

Key Concept: An Exponential **Function** is a function that changes at a **Constant** percentage rate.

Example 3-8: If a population triples each hour, does this represent constant percentage growth?

If so, what is the percentage increase each hour?

Is the population size an exponential function of time?

- **Solution:** The population changes each hour:

$$\text{Population next hour} = 3 \times \text{Current population}$$

Suppose we start with 100 individuals:

$$\text{Initial population} = 100$$

$$\text{Population after 1 hour} = 3 \times 100 = 300$$

$$\text{Population after 2 hours} = 3 \times 300 = 900$$

Let's look at this in terms of growth:

Growth over 1st hour $= 300 - 100 = 200 = 200\%$ increase over 100

Growth over 2nd hour $= 900 - 300 = 600 = 200\%$ increase over 300

The population is growing at a constant percentage rate, 200% each hour. Thus, the population is an exponential function of time.

Now, You Try Yourself:
If a population quadruples each hour, does this represent constant percentage growth? Yes
If so, what is the percentage? Quadrupling represents 300% growth
Is the population size an exponential function of time? Yes, this is an exponential function of time.

Exponential Formulas

The formula for an exponential function *y* of *t* is:

$$y = \text{Initial value} \times \text{Base}^t$$

An exponential function *y* of *t* is characterized by the following property: When *t* increases by 1, to find the new value of *y*, we multiply the current value by the base.

$$y - \text{value for } t + 1 = \text{Base} \times y - \text{value for } t$$

Example 3-9

- **Example:** The value of a certain investment grows according to the rule:

$$\text{Next year's balance} = 1.07 \times \text{Current balance}$$

1. Find the percentage increase each year, and explain why the balance is an exponential function of time.
2. Assume that the original investment is $800. Find an exponential formula that gives the balance in terms of time.
3. What is the balance after 10 years?

- **Solution:**

1. The next year's balance $= 1.07 \times$ this year's balance.

The next year's balance is 107% of this year's balance.

That is an increase of 7% per year. Because the balance grows by the same percentage each year, it is an exponential function of time.

2. Let $B =$ the balance in dollars after t years.

$$B = \text{Initial value} \times \text{Base}^t$$

The initial value = $800. The base = 1.07. This gives the formula:

$$B = 800 \times 1.07^t$$

3. Balance after 10 years $= 800 \times 1.07^{10} = \$1573.72$.

Now, You Try Yourself 3.9:

An investment grows according to the rule:

Next Month's Balance = 1.03 X Current Balance

Find the percentage increase each month. Answer: 3% per month

If the initial investment is $450, find an exponential formula for the balance as a function of time.

If B denotes the balance in dollars after t months, then $B = 450 \times 1.03^t$

What is the balance after two years?

First we need to change two years into equivalent number of months, which in this case is

2 years * 12 months / 1 year = 24 months, then balance after 24 months is:

$$B = 450 \times 1.03^{24} = \$914.76$$

Example 3-10.

- **Example:** Consider the investment from the previous example where the balance B after t years is given by:

$$B = 800 \times 1.07^t \text{ dollars}$$

What is the growth of the balance over the first 10 years? Compare this with the growth from year 40 to year 50.

- **Solution:** The balance after 10 years was \$1573.72.

$$\text{Growth over first 10 years} = \$1573.72 - 800 = \$773.72.$$

To calculate the growth from year 40 to year 50:

$$\text{Balance after 40 years} = 800 \times 1.07^{40} = \$11{,}979.57$$
$$\text{Balance after 50 years} = 800 \times 1.07^{50} = \$23{,}565.62$$

That is an increase of \$23,565.62 − \$11,979.57 = \$11,586.05. That is almost 15 times the growth over the first 10 years.

Try it yourself 3-10:

After t months, an investment has a balance of $B = 400 \times 1.05^t$ *dollars*

Compare the growth rate over the first 20 months with the growth from months 50 to month 70.

Answer: The growth rate over the first 20 months is $1061.32 - 400 =$ **\$661.32**

(0 , 400)

(20 , 1061.32)

Growth from months 50 to 70 is calculated as follows:

(50 , 4586.96)

(70 , 12170.57)

Growth from months 50 to 70 is $12170.57 - 4586.96 =$ **\$7583.61, which is about 11.5 times the earlier growth.**

Definition: Exponential Growth

1. **A quantity grows exponentially when it increases by a <u>constant</u> percentage over a given period.**
2. **If *r* is the percentage growth per period then the base of the exponential function is 1 + r. For exponential growth, the base is always greater <u>than 1.</u>**

$$\boxed{\text{Amount} = \text{Initial value } \times (1 + r)^t}$$

Here, *t* is the number of <u>periods.</u>

3. **Typically, exponential growth starts <u>slowly</u> and then increases <u>rapidly.</u>**

Example 3-11:

- **Example:** U.S. health-care expenditures in 2010 reached 2.47 trillion dollars. In the near term this is expected to grow by 6.5% each year. Assuming that this growth rate continues, find a formula that gives health-care expenditures as a function of time. If this trend continues, what will health-care expenditures be in 2030?

- **Solution:** Let H be the expenditures t years after 2010.

$$H = \text{Initial value } \times (1 + r)^t = 2.47 \times (1 + 0.065)^t.$$

To predict health-care expenditures in 2030, use $t = 20$ in the formula for H:

$$\text{Expenditures in 2030} = 2.47 \times (1 + 0.065)^{20} \text{ trillion dollars.}$$

The above result is about 8.7 trillion dollars.

Try it yourself 3-11:
U.S defense spending was about 719.2 billion dollars in 2010. Since that time, it has grown by about 9.5% each year. Assume that this growth rate continues, and find a formula that gives defense spending as a function of time. What prediction does this formula give for defense spending in 2015?

Answer: If D denotes defense spending in billions of dollars, t years after 2010, $D = 719.2(1+9.5/100)^t = 719.2(1.095)^t$ Billion *dollars*

In 2015, the defense spending will be $D = 719.2(1.095)^5 = 1132.2$ Billion *dollars*

- **Exponential Decay**

1. A quantity decays exponentially when it decreases by a constant percentage over a given period.
2. If r is the percentage decay per period, then the base of the exponential function is $1 - r$. For exponential decay, the base is always less than 1.

Here, t is the number of periods.

3. Typically, exponential decay is rapid first but eventually slows.

$$\boxed{\text{Amount} = \text{Initial value } \times (1 - r)^t}$$

For exponential decay, the base is always less than one
Exponential decay is fast at first and then eventually _________

Example 3.12 on p. 156
After antibiotics are administered, the concentration in the bloodstream declines over time. Suppose that 70 milligrams (mg) of amoxicillin are injected and that the amount of the drug in the bloodstream declines by 49% each hour.

Find an exponential formula that gives the amount of amoxicillin in the bloodstream as a function of time since the injection. Another injection will be required when the level declines to 10 mg. Will another injection be required before five hours?

- **Solution:** Let A be the amount of amoxicillin in the bloodstream after t hours. The base of the exponential function is:

$$1 - r = 1 - 0.49 = 0.51$$

The initial value = 70 mg:

$$A = \text{Initial value } \times (1 - r)^t = 70 \times 0.51^t$$

To find the amount of amoxicillin after 5 hours:

$$A = 70 \times 0.51^5 = 2.4 \text{ mg}$$

The result is about 2.4 mg, which is less than the minimum of 10 mg. Thus, another injection will be needed before 5 hours.

Try it yourself 3-12:
A certain population is initially 4000 and declines by 10% per year. Find an exponential formula that gives the population as a function of time.

Answer: If N denotes the population after t years,
$N = 4000(1-10/100)^t = 4000(0.90)^{\mathrm{t}}$

What will be the population after 10 years? $N = 4000(0.90)^{10} = 1395$

Radioactive decay and half life
***Key concept*- The half-life of a <u>Radioactive</u> substance is the time it takes for half of the substance to decay**

After *h* half-lives, the amount of a radioactive substance remaining is given by the exponential formula: $\mathit{Amount}\ \text{remaining} = \mathit{Initial}\ \text{Amount} * \left(\frac{1}{2}\right)^h$

We can find the amount remaining after *t* years by first expressing *t* in terms of half-lives and then using the formula above.

Example: Pu-239 has a half life of 24,000 years. We start with 100 gm.

So, after 24000 years we would get _<u>50</u> gm.

How much material is left after 36,000 years?

First we need to calculate the amount of half lives in 36000 years.

$$36000\, years * \frac{halflife}{24000\, years} = 1.5\ halflife$$

Next, we will use the formula $Amount$ remaining $= Initial$ Amount $* \left(\frac{1}{2}\right)^h$

So, the amount remaining after 36000 years = $100 * \left(\frac{1}{2}\right)^{1.5} = 35.4\ grams$

Try it yourself 3-14:
A certain radioactive substance has a half-life of 450 years. If there are initially 20 grams, find a formula for the exponential function that gives the amount remaining after h half lives.

Solution: $A = 20 * \left(\frac{1}{2}\right)^h$ Where A is the amount remaining after h half lives.

How much remains after 720 years? Round your answer to one decimal place.

Solution: First we need to calculate the amount of half lives in 720 years

$$720\, years * \frac{halflife}{450\, years} = 1.6\ halflife$$

$$A = 20 * \left(\frac{1}{2}\right)^{1.6} = 6.6\, grams \text{ remaining}$$