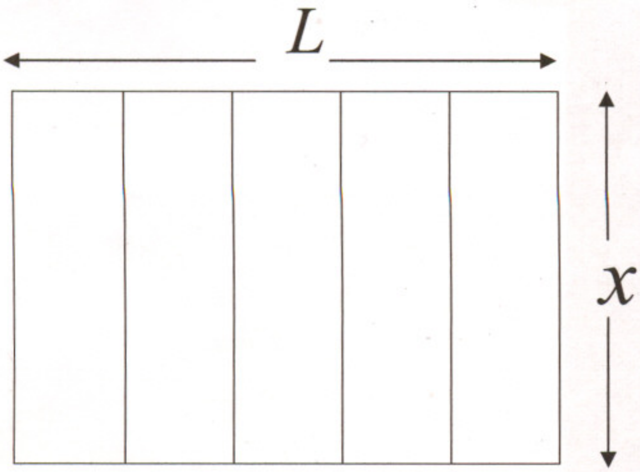
Name: Solution

MA 160 Dr. Katiraie (100 points) Test #3 Spring 2013

Show all of your work on the test paper. All of the problems must be solved symbolically using Calculus. You may use your calculator to confirm your answer, but full credit is not given unless the answer follows from the symbolic work shown.

1. A farmer wants to enclose a rectangular area and then divide it into **five** smaller rectangular regions by putting up fencing parallel to one side of the rectangle. The total amount of fencing available is 2400 feet. An illustration is shown below.

A) Express the total area of the enclosed region as a function of x and simplify your answer. (4 points)



$6X + 2L = 2400$

$2L = 2400 - 6X$

$L = 1200 - 3X$

$\text{Area} = x(1200 - 3X)$

$\boxed{\text{Area} = 1200X - 3X^2 \text{ (feet}^2)}$

B) State the Domain of the above function. (Hint: what are the possible values of x) (2 points)

$1200 - 3X > 0$

$X < 400$

$\boxed{0 < X < 400 \text{ feet}}$

2. If f is a differentiable function of x and $g(x) = \sqrt[3]{x}\, f(x) = x^{1/3} f(x)$

(a) Find an expression for the derivative of $g(x)$ in terms of $f(x)$ *and* $f'(x)$. (4 points)

$\boxed{g'(x) = \frac{1}{3}X^{-2/3} f(x) + X^{1/3} f'(x)}$

(b) If it is known that $f(8) = 12$ *and* $f'(8) = 5$, (4 points)

Find $g'(8)$.

$g'(8) = \frac{1}{3}(8)^{-\frac{2}{3}} f(8) + (8)^{1/3} f'(8)$

$= \frac{1}{3}(8)^{-2/3}(12) + (8)^{1/3}(5) = \boxed{11}$

3. The radioactive substance Bismuth-210 has a half-life of 5.0 days.

(a) A sample originally has a mass of 5000 mg. Find a formula for the mass remaining after t days (Hint: find an exponential function of the type $A(t) = Ce^{kt}$ that models the above situation)

(4 points)

$$A(t) = 5000e^{Kt}$$

$$2500 = 5000e^{K(5)}$$

$$\frac{1}{2} = e^{5K}$$

$$K = \frac{\ln(1/2)}{5} \implies \boxed{A(t) = 5000e^{\frac{\ln(\frac{1}{2})}{5}t}}$$

(b) Find the mass remaining after 5 days.

(3 points)

$$A(5) = 5000e^{\frac{\ln(\frac{1}{2})}{5}5} = \boxed{2500 \text{ mg}}$$

(c) When is the mass reduced to 1000 mg?

(4 points)

$$1000 = 5000e^{\frac{\ln(1/2)}{5}t}$$

$$\frac{1}{5} = e^{\frac{\ln(\frac{1}{2})}{5}t}$$

$$\ln\frac{1}{5} = \frac{\ln(\frac{1}{2})}{5}t \qquad \boxed{t = 11.61 \text{ days}}$$

4. Differentiate the following function

$$y = \frac{4x + e^{-2x}}{5x - 3}$$

(6 points)

$$y' = \frac{(4 - 2e^{-2x})(5x-3) - 5(4x + e^{-2x})}{(5x-3)^2}$$

$$= \frac{\cancel{20x} - 12 - 10xe^{-2x} + 6e^{-2x} - \cancel{20x} - 5e^{-2x}}{(5x-3)^2} = \boxed{\frac{-10xe^{-2x} + 1e^{-2x} - 12}{(5x-3)^2}}$$

5. Differentiate the following function

$f(x) = (5x-3)^7(2x-3)^4$ (6 points)

$$f'(x) = 7(5x-3)^6(5)(2x-3)^4 + 4(2x-3)^3 \cdot 2(5x-3)^7$$

$$= 35(5x-3)^6(2x-3)^4 + 8(2x-3)^3(5x-3)^7$$

$$= (2x-3)^3(5x-3)^6\left(35(2x-3) + 8(5x-3)\right)$$

$$\boxed{= (2x-3)^3(5x-3)^6(110x - 129)}$$

$$\begin{array}{r} 1 \\ 35 \\ 3 \\ \hline 105 \\ -24 \end{array}$$

6. Let $f(x) = 3x^2 + \dfrac{4}{x}$ $x > 0$ (10 points)

a) Find $f'(x)$

$$f(x) = 3x^2 + \frac{4}{x} = 3x^2 + 4x^{-1}$$

$$f'(x) = 6x - 4x^{-2} = \boxed{6x - \frac{4}{x^2}}$$

b) Find any critical numbers of f(x) on the given domain.

$$f'(x) = \frac{6x^3 - 4}{x^2}$$

$$x^3 = \frac{4}{6} \Rightarrow x = \sqrt[3]{\frac{4}{6}} \cong \boxed{0.8736}$$

$$x^2 = 0 \Rightarrow \cancel{\boxed{x = 0}}$$ Not in Domain

$$\left\{\cancel{x = 0} \ \& \ \boxed{x = 0.8736}\right\}$$

c) Use the First or Second Derivative Test to determine if the function f(x) has a relative maximum or a relative minimum at any critical number you found in part (b). **State your answer clearly, that is, does the function have a relative maximum or does it have a relative minimum at any critical number you found in part (b)?**

Graph $Y_1 = 3x^2 + \dfrac{4}{x}$

$f' = \ominus$, $f' = \ominus$, $f' = \oplus$

$x = 0$ $\quad$ $x = 0.8736$

$\searrow$ $\quad$ $\searrow$ $\quad$ $\nearrow$

Relative Minimum is 6.868 occurs at $x = 0.8736$

7. Let $f(x)=\ln(e^{5x}+2x)$. Find the equation of the tangent line to this function when x = 0 on the graph. Write your answer using exact values, not approximations. (8 points)

$f(x)=\ln(e^{5x}+2x)$

$f'(x)=\frac{1}{e^{5x}+2x}(5e^{5x}+2)\Big|_{\text{at } x=0} \Rightarrow m=\frac{5e^{0}+2}{1}=7$

$f(0)=\ln(e^{0}+0)=\ln(1)=0$

$y-0=7(x-0) \Rightarrow \boxed{y=7x}$

8. For what values or values of x does the function $f(x)=x^4\ln(x)$, $x>0$, have a horizontal tangent line? Write your answer using exact values, not approximations. (8 points)

$f(x)=x^4\ln(x)$ $\quad f'(x)=4x^3\ln x+x^4\frac{1}{x}=4x^3\ln x+x^3$

let $f'(x)=0 \Rightarrow 4x^3\ln x+x^3=x^3(4\ln x+1)=0$ $\cancel{x=0}$

$4\ln x+1=0$

$\ln x=\frac{-1}{4} \Rightarrow \boxed{x=e^{\frac{-1}{4}}}$

9. When a company produces and sells x units of a certain product each week, its profit (in dollars) that week is $P(x)=-.02x^2+30x-5000$. The company is increasing production each week so that the production level t weeks from now will be $x=500+25t$ (9 points)

(a) Find the marginal profit $\frac{dP}{dx}$

$P(x)=-.02x^2+30x-5000$ $\Rightarrow \frac{dP}{dx}=-0.04x+30$

(b) Find the time rate of change of profit $\frac{dP}{dt}$ $=\frac{dP}{dx}\cdot\frac{dx}{dt}=(-0.04x+30)(25)$

$\boxed{=-1x+750}$

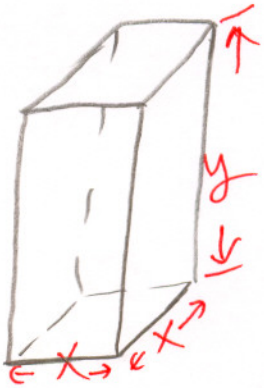
(c) How fast (with respect to time) are profits changing when t = 10? **Write a sentence in everyday language to explain the meaning of your answer. Be sure to use appropriate units.**

$\frac{dP}{dt}=-1(500+25(10))+750=\boxed{0}$

After 10 weeks, Profits are changing at a rate of \$0 per week

i.e the profit is Not changing.

10. A large closed rectangular storage box with a square base is to be constructed using two different types of wood. The base is made of wood costing $6 per square foot and the top and sides are made of wood costing $4 per square foot. Suppose that the amount available to spend is $800.
Find the dimensions of the box of greatest volume that can be constructed.
(please round your answers to two decimal places) (10 points)



$6x^2 + 4x^2 + 4(4xy) = 800$

$10x^2 + 16xy = 800$

$y = \frac{800 - 10x^2}{16x}$

$V = x^2\left(\frac{800 - 10x^2}{16x}\right) = 50x - \frac{10}{16}x^3 = 50x - \frac{5}{8}x^3$

$V' = 50 - \frac{15}{8}x^2 = 0 \Rightarrow x^2 = \frac{50}{(15/8)} \Rightarrow x = \sqrt{\frac{50}{(15/8)}} = \boxed{5.16 \text{ ft}}$

$y = \frac{800 - 10(5.16)^2}{16(5.16)} = \boxed{6.46 \text{ ft}}$

11. A manufacturer of bicycle helmets estimates that the demand (price) of each helmet is $p = 31 - 0.003x$ (8 points)

(a) Write the revenue function R(x) of the product.

$R(x) = x(31 - 0.003x) = \boxed{31x - 0.003x^2}$

(b) Suppose the total cost of producing x helmets is given by $C(x) = 6200 + 7.3x + 0.002x^2$
Find and simplify the profit function P(x)

$P(x) = (31x - 0.003x^2) - (6200 + 7.3x + 0.002x^2)$

$P(x) = \boxed{-0.005x^2 + 23.7x - 6200}$

(c) How many helmets must the company produce and sell in order to maximize profit?

$P' = -0.01x + 23.7 = 0 \Rightarrow x = \frac{23.7}{0.01} = \boxed{2370 \text{ Helmets}}$

(d) What price per helmet must be charged in order to maximize profit?

$\text{Price} = 31 - 0.003x$

$= 31 - 0.003(2370) = \boxed{\$23.89}$

12. A fast food restaurant is establishing its inventory policy for ordering frozen french fries. In the coming year, they expect to sell 800 pounds of french fries. It costs \$16 to place an order and the carrying costs for a year are \$4 per pound based on the average amount in storage. (10 points)

(a) Determine the economic order quantity for the french fries, that is, the order size which minimizes the inventory cost.

let x = order size

r = number of orders per year

$$xr = 800 \Rightarrow r = \frac{800}{x}$$

$$\text{Cost} = 16r + 4\left(\frac{x}{2}\right) = 16r + 2x$$

$$\text{Cost} = 16\left(\frac{800}{x}\right) + 2x = 12800x^{-1} + 2x$$

$$\text{Cost}' = -12800x^{-2} + 2 = 0 \Rightarrow \frac{12800}{x^2} = 2 \Rightarrow x^2 = 6400$$

$$\boxed{x = 80 \text{ lbs}}$$

(b) How many times a year should the fast food restaurant order french fries in order to minimize inventory cost?

$$r = \frac{800}{x} = \frac{800}{80} = \boxed{10 \text{ orders}}$$