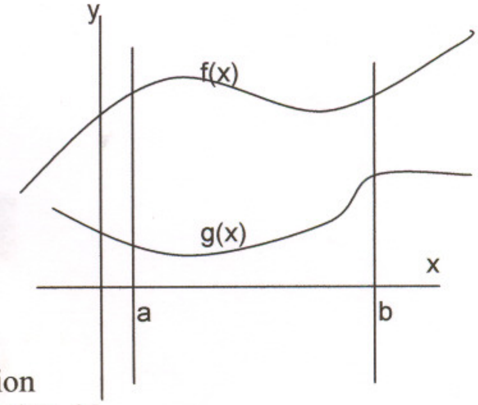
MA 160
Section 6.1

Areas Between Curves *Solutions by Dr. Katiraie*

1. Let f and g be functions whose graphs are shown.



(a) Write an integral which represents the area of the region between the graph of f(x) and the x-axis on the interval [a, b].

$$\int_a^b f(x)\,dx$$

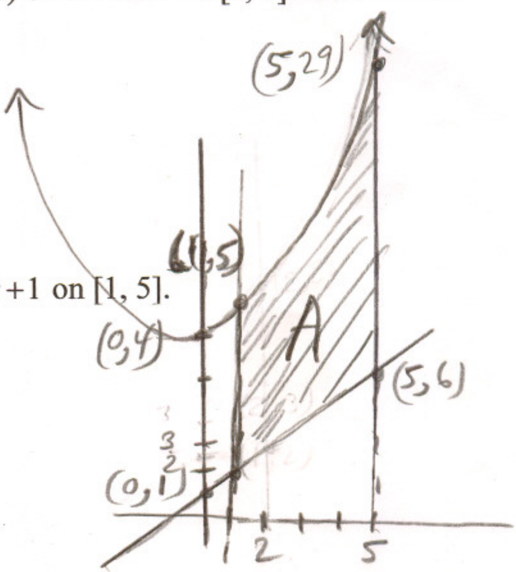
(b) Write an integral which represents the area of the region between the graph of g(x) and the x-axis on the interval [a, b].

$$\int_a^b g(x)\,dx$$

(c) Combine the two integrals in parts (a) and (b) to write an expression which represents the area of the region between the graphs of f(x) and g(x) on the interval [a, b]. Rewrite this as a single integral.

$$\int_a^b \left(f(x)-g(x)\right)dx$$

2.
(a) Sketch a graph of the functions $y=x^2+4$ and $y=x+1$ on [1, 5].



(b) Set up an integral which represents the area of the region between $y=x^2+4$ and $y=x+1$ on [1, 5].

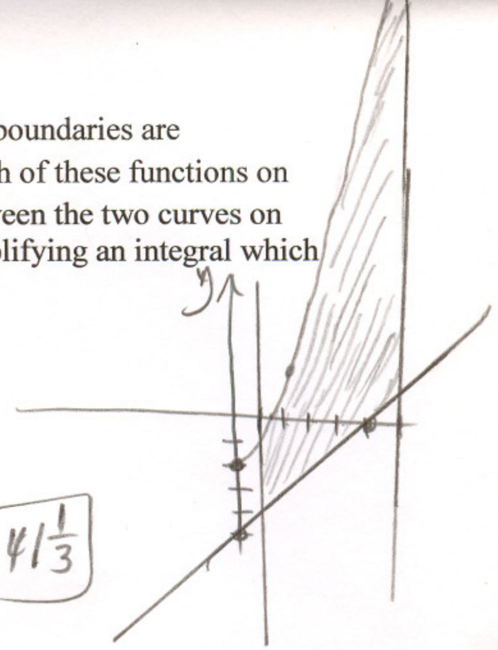
$$\text{Area} = \int_1^5 \left((x^2+4)-(x+1)\right)dx = \int_1^5 (x^2+4-x-1)\,dx$$
$$= \int_1^5 (x^2-x+3)\,dx$$

(c) Evaluate the integral that you set up in part (b).

$$\frac{x^3}{3}-\frac{x^2}{2}+3x\Big|_1^5 = \left(\frac{5^3}{3}-\frac{5^2}{2}+3(5)\right)-\left(\frac{1^3}{3}-\frac{1^2}{2}+3(1)\right)$$
$$= 41.\overline{333} = \boxed{41\tfrac{1}{3}}$$

OVER →

3. Suppose that the region in #2 is ***lowered*** 6 units so that the new boundaries are $y = x^2 + 4 - 6 = x^2 - 2$ and $y = x + 1 - 6 = x - 5$. Sketch the graph of these functions on [1, 5]. Do you think that the value of the area of the region between the two curves on [1, 5] will change? Confirm your answer by setting up and simplifying an integral which represents the area of this new region.



$$\int_1^5 (x^2-2)-(x-5)\,dx = \int_1^5 (x^2-2-x+5)\,dx$$

$$=\int_1^5 (x^2-x+3)\,dx$$

$$= \frac{x^3}{3}-\frac{x^2}{2}+3x\Big|_1^5 = \left(\frac{5^3}{3}-\frac{5^2}{2}+3(5)\right)-\left(\frac{1^3}{3}-\frac{1^2}{2}+3(1)\right)=\boxed{41\tfrac{1}{3}}$$

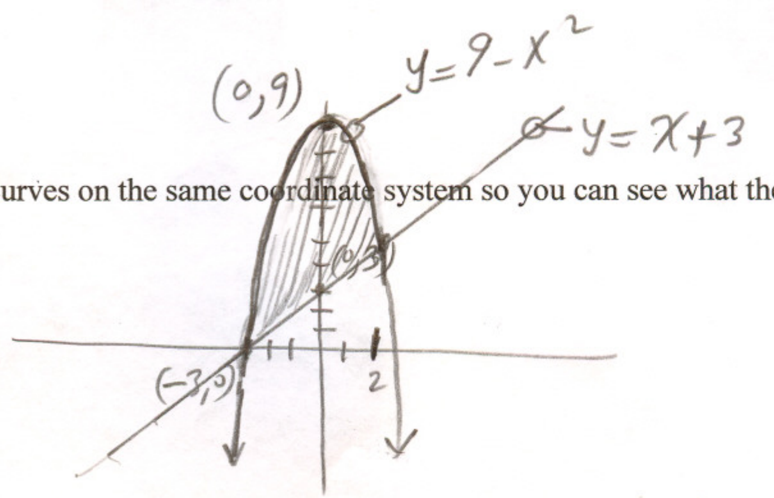
4. In general, what must be true about functions f and g so that $\int_a^b [f(x)-g(x)]\,dx$ represents the value of the area of the region between the graphs of f and g on the interval [a, b]?

f(x) must be above g(x) i.e $f(x) > g(x)$

5. In this problem, you are going to find the area of the region between the graphs of the functions $y = 9 - x^2$ and $y = x + 3$. To solve this problem,

(a) Since no boundary points are given, you must first find the points of intersection of the graphs of the functions. To do this, set the two functions equal and solve for x.

$$9-x^2 = x+3$$
$$-9 \qquad\qquad -9$$
$$0 = x^2+x-6$$
$$0 = (x+3)(x-2)$$
$$x=-3 \qquad x=2$$



(b) Sketch a graph of the two curves on the same coordinate system so you can see what the region looks like.

(c) Set up an integral which represents the value of the area of the given region and then evaluate this integral to answer the question.

$$\int_{-3}^{2}\left((9-x^2)-(x+3)\right)dx = \int_{-3}^{2}(9-x^2-x-3)\,dx = \int_{-3}^{2}(-x^2-x+6)\,dx$$

$$= -\frac{x^3}{3}-\frac{x^2}{2}+6x\Big|_{-3}^{2} = \left(\frac{-2^3}{3}-\frac{2^2}{2}+6(2)\right)-\left(\frac{-(-3)^3}{3}-\frac{(-3)^2}{2}+6(-3)\right)\approx \boxed{20.833}$$

1. **Consumer Surplus**

The demand for a product, in dollars, is $p = 1200 - 0.2q - 0.0001q^2$
Find the consumer surplus when the sales level is 500.

"The consumer surplus of a product when Q units are sold at a price D(Q) is given by $\int_0^Q [D(q) - P]\,dq$ "

Since the number of products sold is $Q = 500$, the corresponding price is $P = 1200 - 0.2(500) - 0.0001(500)^2 = 1075$

Therefore, by using the definition of consumer surplus;

$$\text{Consumer Surplus} = \int_0^{500} \left[1200 - 0.2q - 0.0001q^2 - 1075\right] dq$$

$$= \int_0^{500} (125 - 0.2q - 0.0001q^2)\,dq$$

$$= \left(125q - 0.2\frac{q^2}{2} - \frac{0.0001q^3}{3}\right)\Big|_0^{500}$$

$$= 125(500) - \frac{0.2}{2}(500)^2 - \frac{0.0001}{3}(500)^3 - 0$$

$$= \boxed{\$\ 33,333.33}$$

2. **Producer Surplus**

An electronic manufacturer estimates that the supply function for its digital clocks is

$$S(q) = 5.4 + 0.001q^{1.2} \text{ dollars.}$$

Find the producer surplus when the number of clocks sold is 2000.

" The Producer Surplus of a product when Q units are sold at a price $P = S(Q)$ is given by

$$\int_0^Q (P - S(q))\, dq$$ "

The number of clocks produced is $Q = 2000$ and the corresponding price is $S(2000) = 5.4 + 0.001(2000)^{1.2} \approx 14.55$

using the above definition of Producer Surplus;

$$\text{Producer Surplus} = \int_0^{2000} (P - S(q)\, dq$$

$$= \int_0^{2000} \left(14.55 - (5.4 + 0.001\, q^{1.2})\right) dq$$

$$= \int_0^{2000} (14.55 - 5.4 - 0.001\, q^{1.2})\, dq$$

$$= \int_0^{2000} (9.15 - 0.001\, q^{1.2})\, dq$$

$$= \left(9.15q - \frac{0.001\, q^{2.2}}{2.2}\right)\Big|_0^{2000}$$

$$= 9.15(2000) - \frac{0.001}{2.2}(2000)^{2.2} - 0 = \boxed{\$9985.36}$$