Section 2.7

Mathematical Models: Constructing Functions

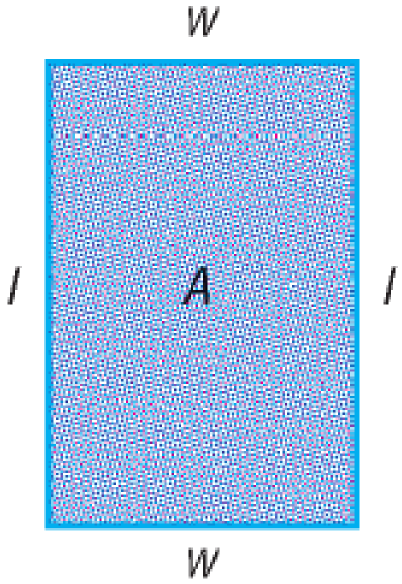
OBJECTIVE 1

Construct and Analyze Functions

EXAMPLE

Area of a Rectangle with Fixed Perimeter

The perimeter of a rectangle is 300 feet. Express its area A as a function of the length l of a side.



$$A = lw = l(150 - l)$$

$$A(l) = l(150 - l)$$

EXAMPLE Economics: Demand Equations

In economics, revenue R, in dollars, is defined as the amount of money received from the sale of a product and is equal to the unit selling price p, in dollars, of the product times the number x of units actually sold. That is,

$$R = xp$$

In economics, the Law of Demand states that p and x are related: As one increases, the other decreases. Suppose that p and x are related by the following **demand equation**:

$$p = -\frac{1}{8}x + 24, \quad 0 \le x \le 192$$

Express the revenue R as a function of the number x of units sold.

$$R(x) = xp = x\left(-\frac{1}{8}x + 24\right) = -\frac{1}{8}x^2 = 24x$$

EXAMPLE

Finding the Distance from the Origin to a Point on a Graph

Let $P = (x, y)$ be a point on the graph of $y = x^2 - 1$.

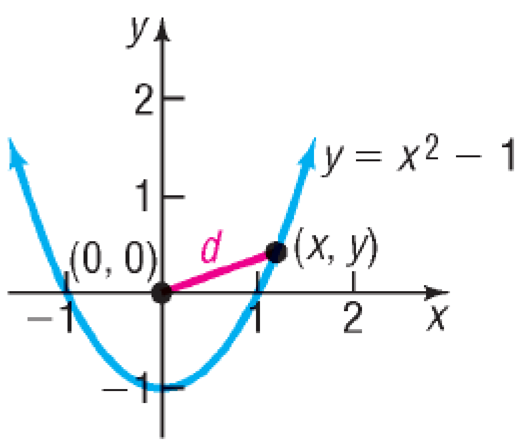
(a) Express the distance d from P to the origin O as a function of x.

(b) What is d if $x = 0$?

(c) What is d if $x = 1$?

(d) What is d if $x = \frac{\sqrt{2}}{2}$?

(e) Use a graphing utility to graph the function $d = d(x)$, $x \geq 0$. Rounded to two decimal places, find the value(s) of x at which d has a local minimum. [This gives the point(s) on the graph of $y = x^2 - 1$ closest to the origin.]



$$d = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

$$d(x) = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^4 - x^2 + 1}$$

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Finding the Distance from the Origin to a Point on a Graph

Let $P = (x, y)$ be a point on the graph of $y = x^2 - 1$.

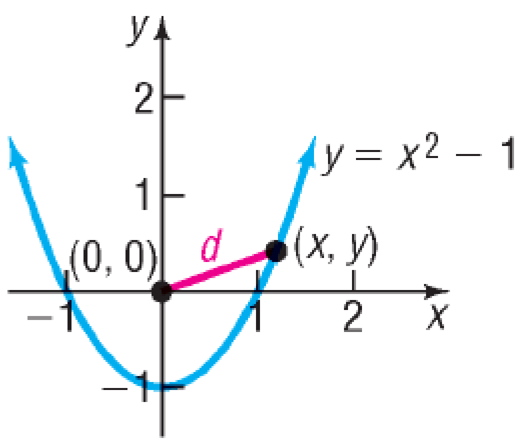
(a) Express the distance d from P to the origin O as a function of x.

(b) What is d if $x = 0$? $d(0) = \sqrt{1} = 1$

(c) What is d if $x = 1$? $d(1) = \sqrt{1 - 1 + 1} = 1$

(d) What is d if $x = \frac{\sqrt{2}}{2}$? $d\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^2 + 1} = \frac{\sqrt{3}}{2}$

(e) Use a graphing utility to graph the function $d = d(x)$, $x \geq 0$. Rounded to two decimal places, find the value(s) of x at which d has a local minimum. [This gives the point(s) on the graph of $y = x^2 - 1$ closest to the origin.]



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EXAMPLE

Finding the Distance from the Origin to a Point on a Graph

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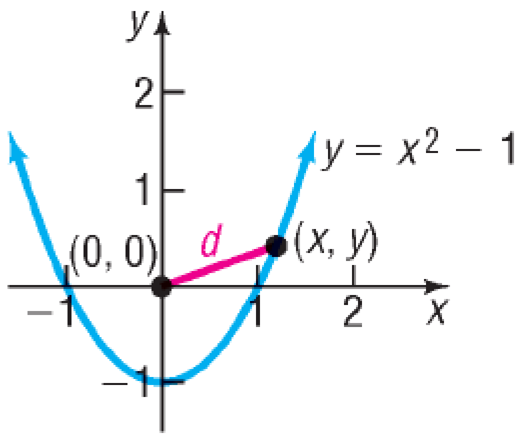
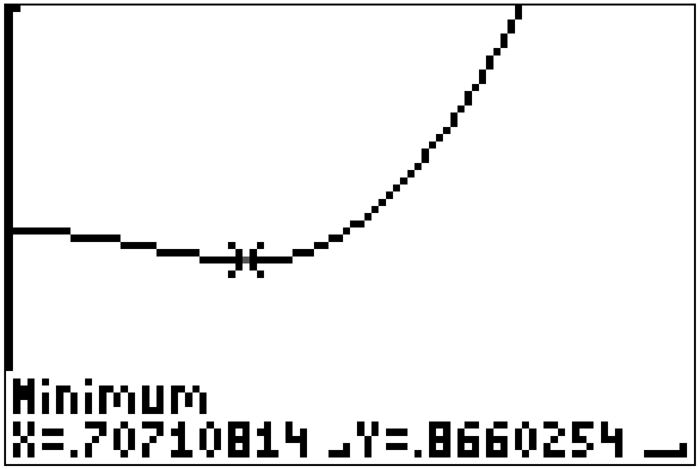
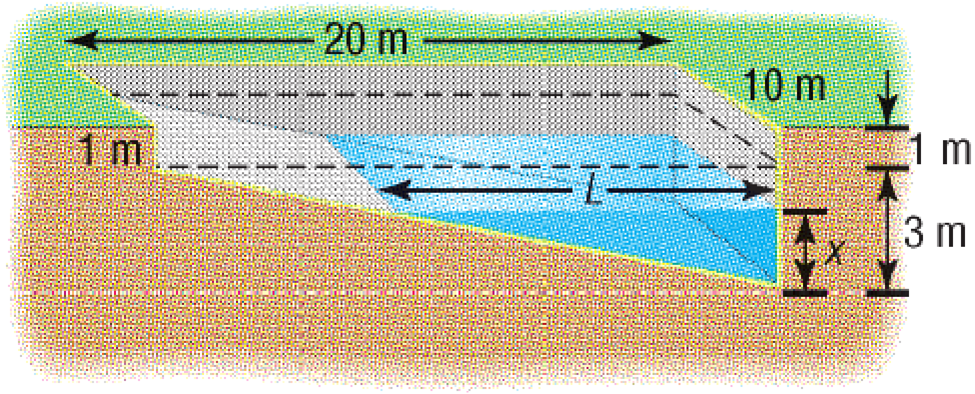




EXAMPLE Filling a Swimming Pool

A rectangular swimming pool 20 meters long and 10 meters wide is 4 meters deep at one end and 1 meter deep at the other. Figure 81 illustrates a cross-sectional view of the pool. Water is being pumped into the pool to a height of 3 meters at the deep end.



(a) Find a function that expresses the volume V of water in the pool as a function of the height x of the water at the deep end.

(b) Find the volume when the height is 1 meter.

(c) Find the volume when the height is 2 meters.

(d) At what height is the volume 20 cubic meters? 100 cubic meters?

EXAMPLE Area of a Rectangle

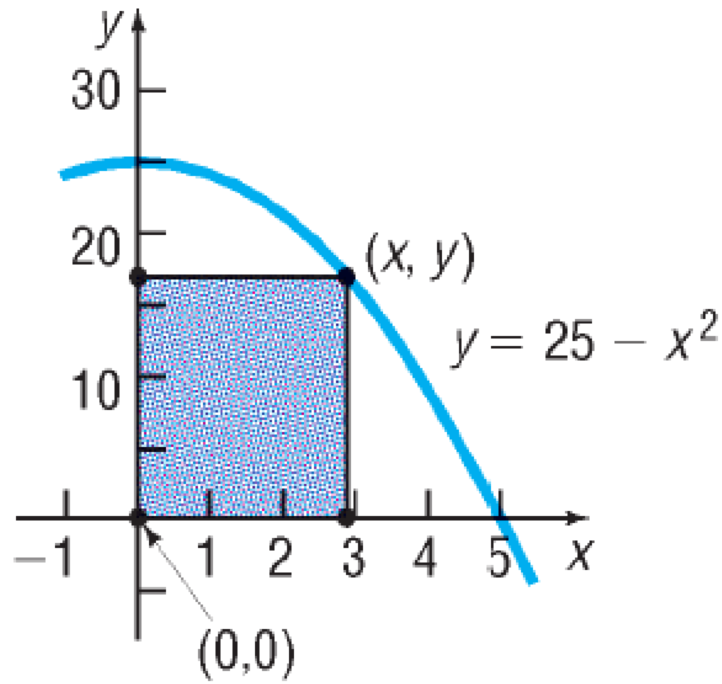
A rectangle has one corner on the graph of $y = 25 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis.

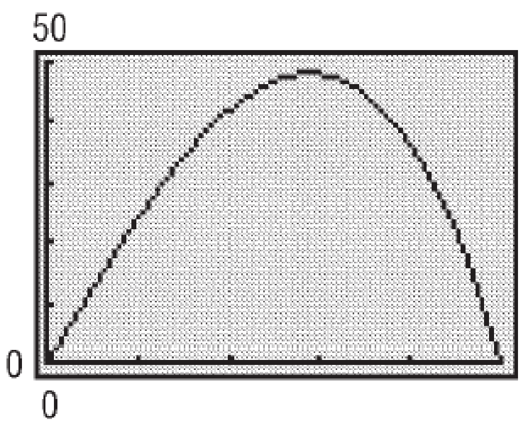
(a) Express the area A of the rectangle as a function of x.

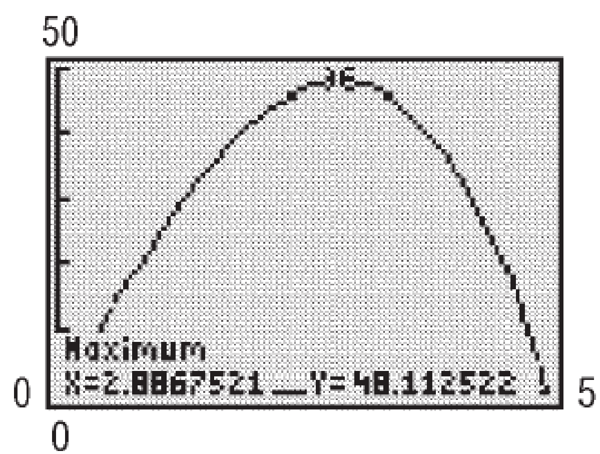
(b) What is the domain of A?

(c) Graph $A = A(x)$.

(d) For what value of x is the area largest?







EXAMPLE Making a Playpen

A manufacturer of children's playpens makes a square model that can be opened at one corner and attached at right angles to a wall or, perhaps, the side of a house. If each side is 3 feet in length, the open configuration doubles the available area in which the child can play from 9 square feet to 18 square feet. See Figure 85.

Now suppose that we place hinges at the outer corners to allow for a configuration like the one shown in Figure 86.

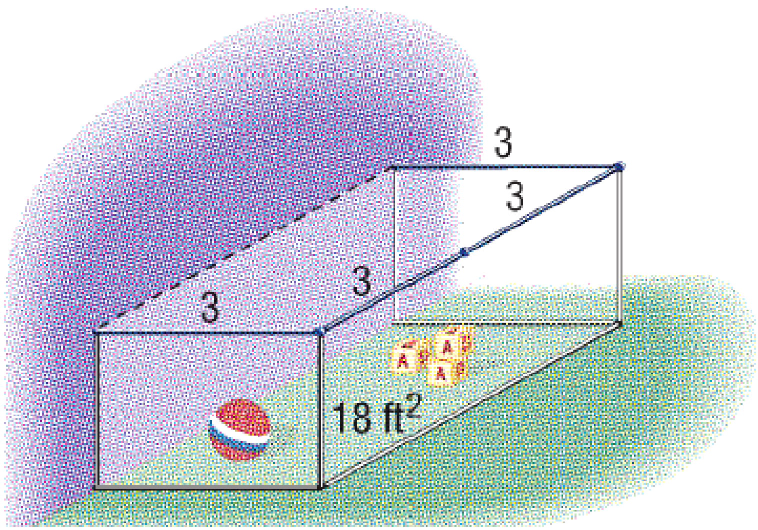


Figure 85

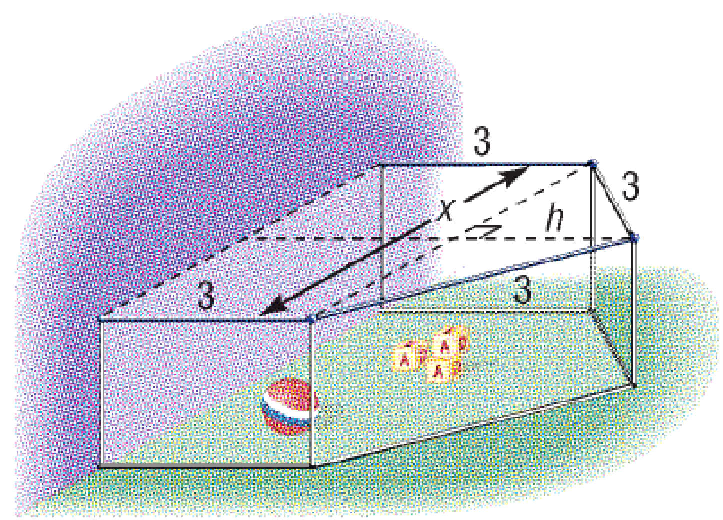


Figure 86

EXAMPLE Making a Playpen

(a) Express the area A of this configuration as a function of the distance x between the two parallel sides.

(b) Find the domain of A.

(c) Find A if $x = 5$.

(d) Graph $A = A(x)$. For what value of x is the area largest? What is the maximum area?

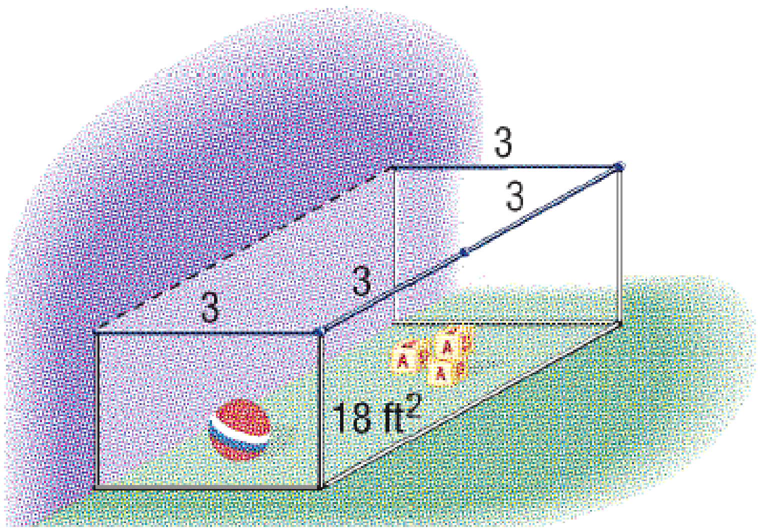


Figure 85

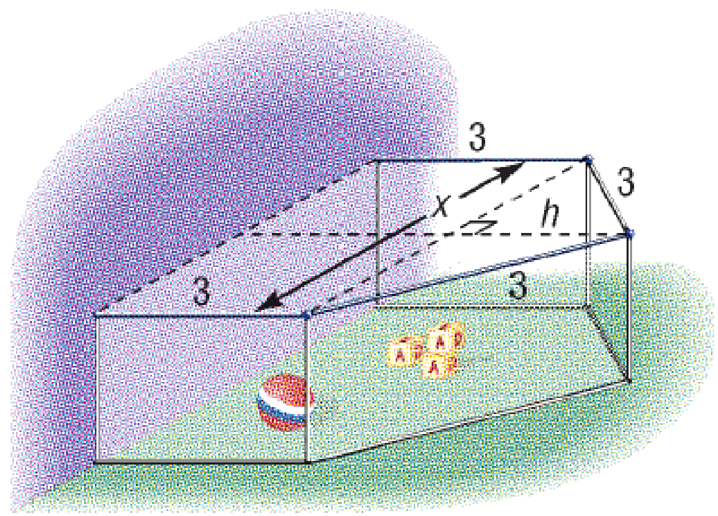


Figure 86