Section 3.1

Quadratic Functions and Models

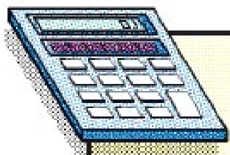
A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$

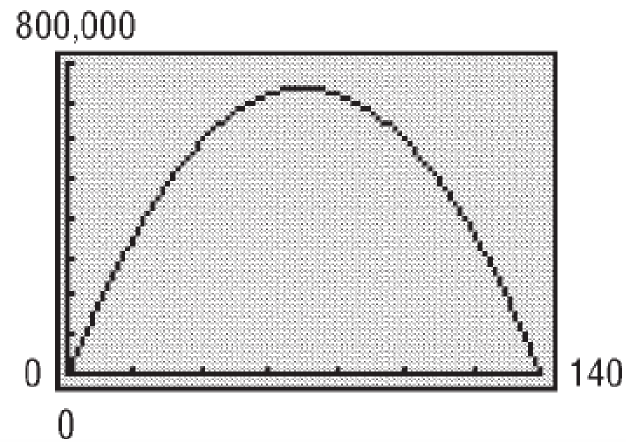
suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price p (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number x of calculators sold and the price p per calculator may be approximated by the linear equation

$$x = 21{,}000 - 150p$$



Price per Calculator, *p* (Dollars)	Number of Calculators, *x*
60	11,100
65	10,115
70	9,652
75	8,731
80	8,087
85	7,205
90	6,439

$$\begin{aligned} R &= xp \\ R(p) &= (21{,}000 - 150p)p \\ &= -150p^2 + 21{,}000p \end{aligned}$$



Then the revenue R derived from selling x calculators at the price p per calculator is equal to the unit selling price p of the product times the number x of units actually sold.

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's second law of motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.

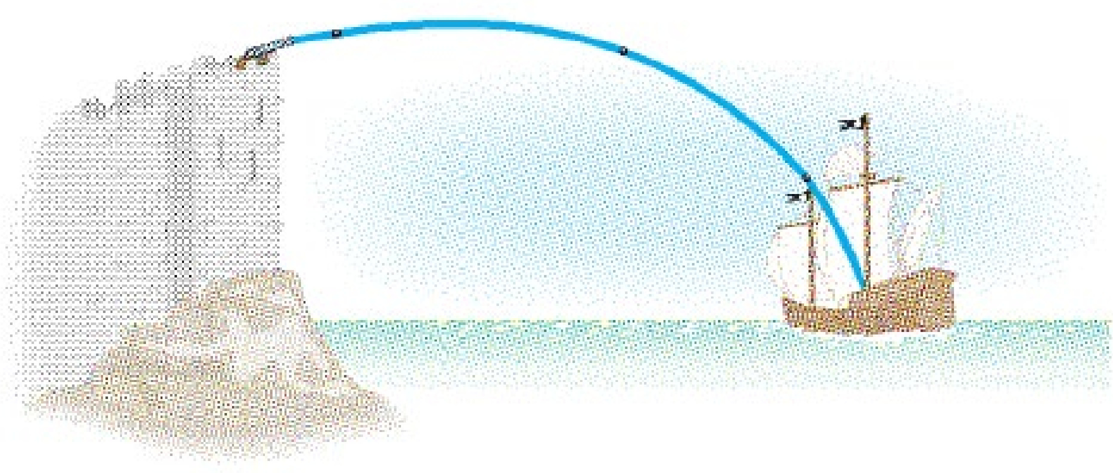
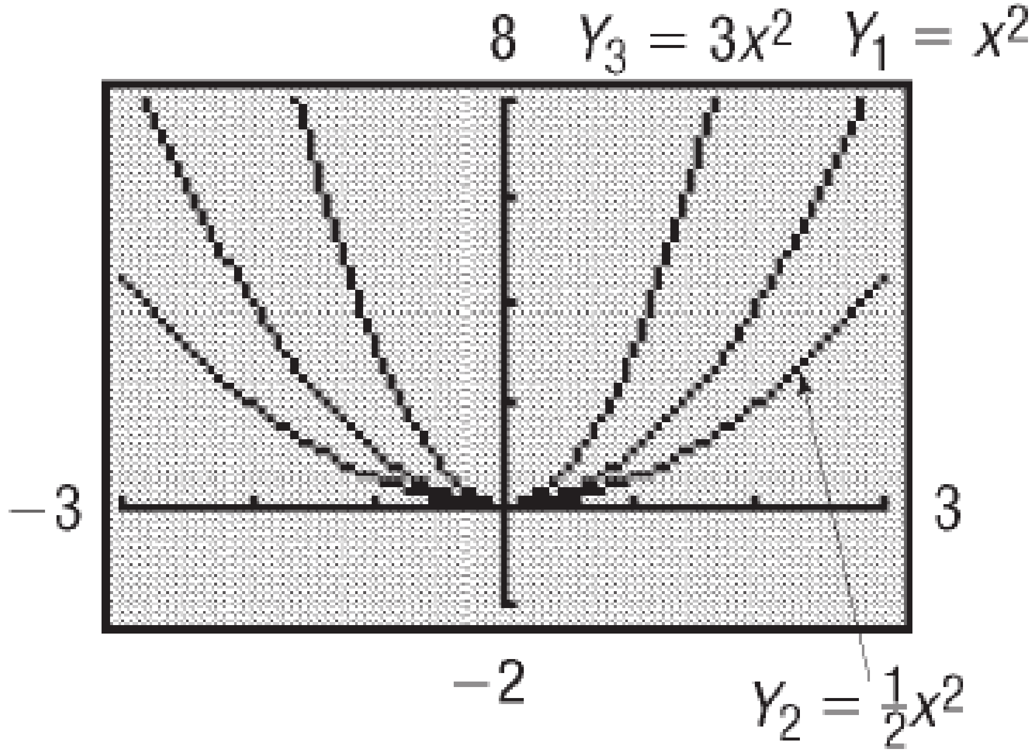


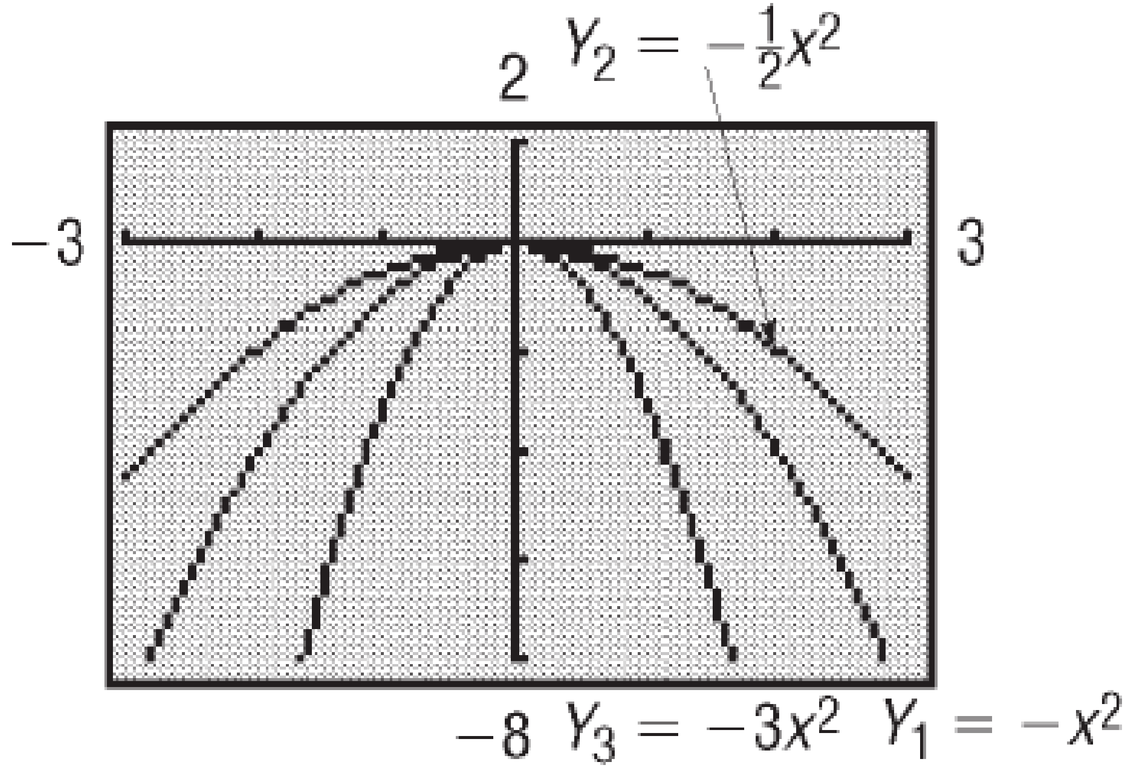
Figure 2
Path of a cannonball

OBJECTIVE 1

1 Graph a Quadratic Function Using Transformations

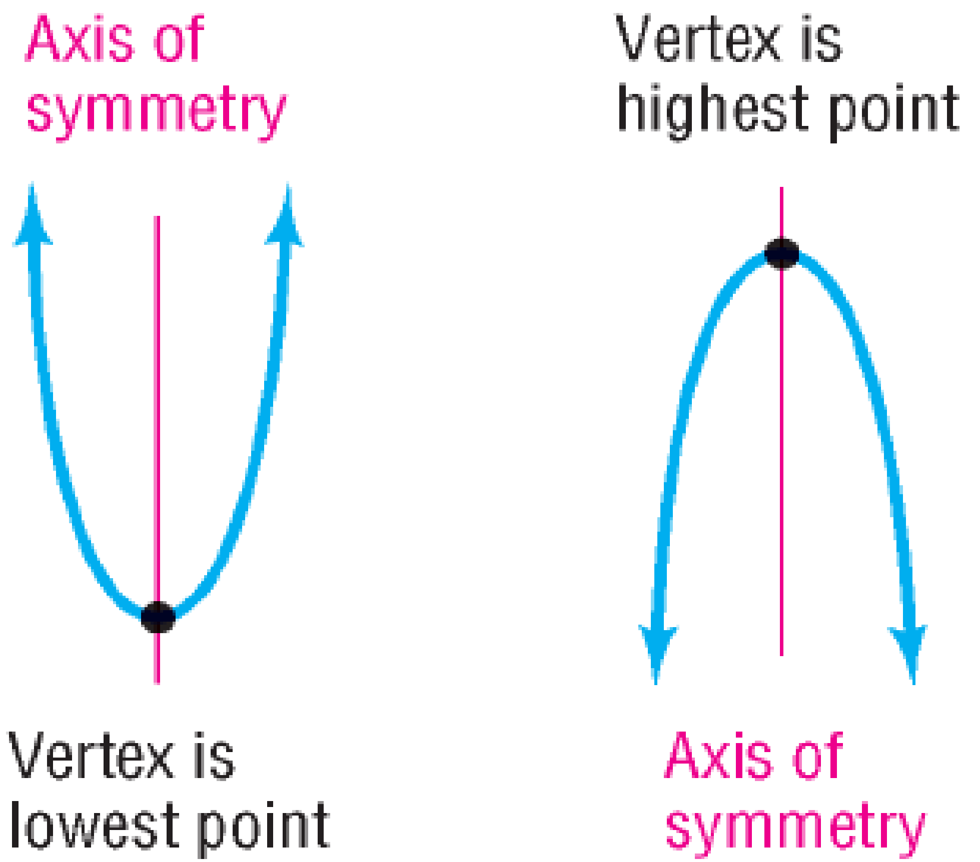


$$f(x) = ax^2, a > 0, \text{ for } a = 1, a = \frac{1}{2}, \text{ and } a = 3.$$



$$f(x) = ax^2 \text{ for } a < 0.$$

Graphs of a quadratic function,
$f(x) = ax^2 + bx + c, a \neq 0$



(a) Opens up

$a > 0$

(b) Opens down

$a < 0$

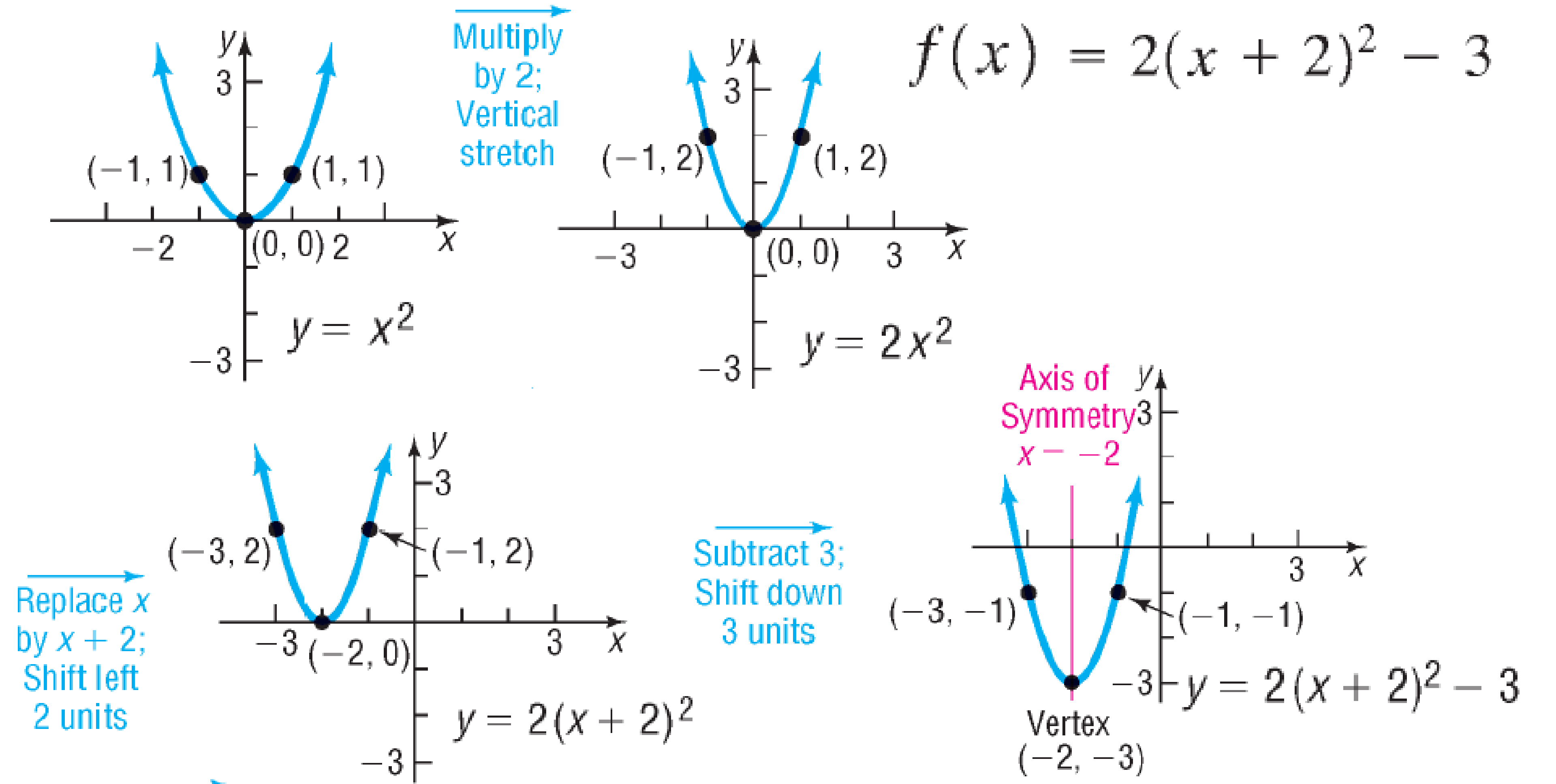
EXAMPLE

Graphing a Quadratic Function Using Transformations

Graph the function $f(x) = 2x^2 + 8x + 5$.

Find the vertex and axis of symmetry.

$$f(x) = 2(x + 2)^2 - 3$$



If $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$, then

$$f(x) = ax^2 + bx + c = a(x - h)^2 + k$$

OBJECTIVE 2

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

Properties of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

Vertex $= \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ Axis of symmetry: the line $x = -\frac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point.

EXAMPLE

Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = 2x^2 - 3x + 2$. Does it open up or down?

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

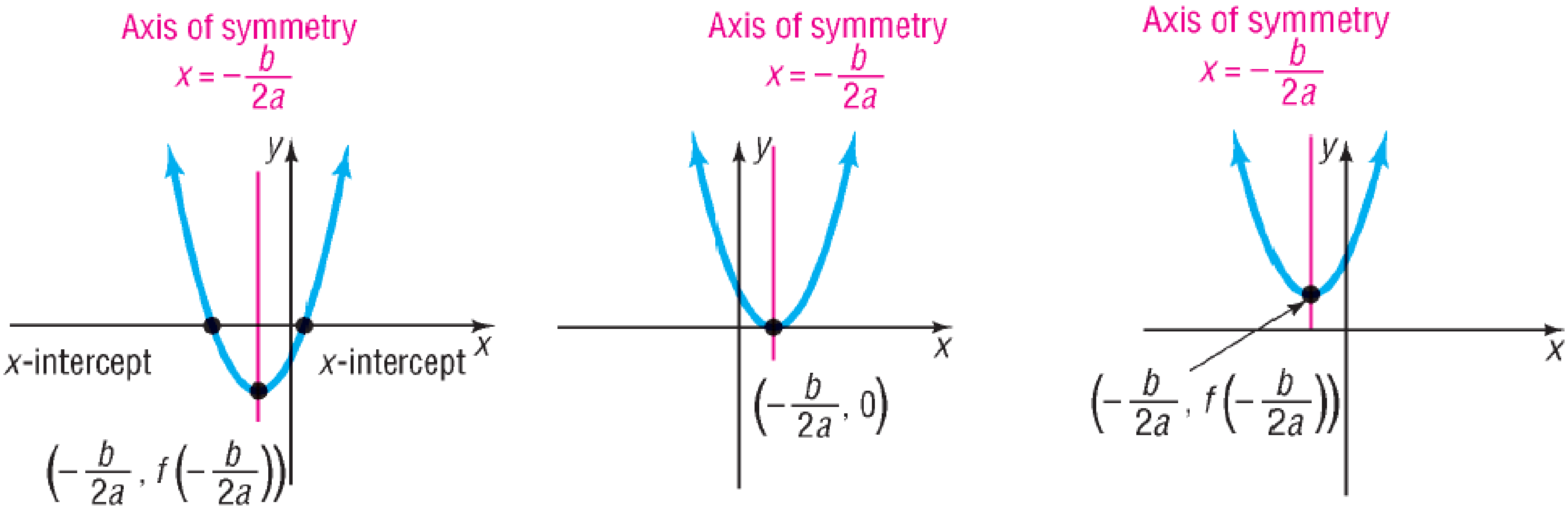
$$\text{Vertex is } \left(\frac{3}{4}, \frac{7}{8}\right)$$

OBJECTIVE 3

3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The x-intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts and so will cross the x-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept and touches the x-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept and so will not cross or touch the x-axis.



(a) $b^2 - 4ac > 0$

Two x-intercepts

(b) $b^2 - 4ac = 0$

One x-intercept

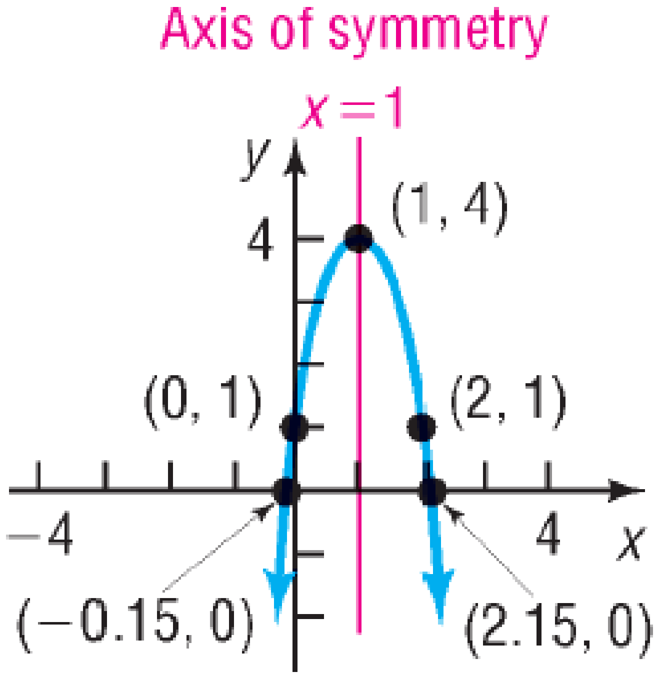
(c) $b^2 - 4ac < 0$

No x-intercepts

EXAMPLE

Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

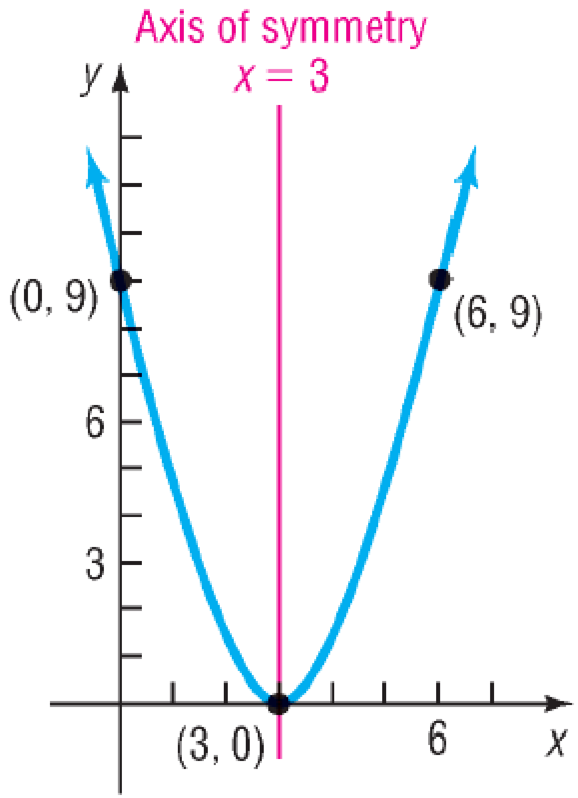
Graph $-3x^2+6x+1$ by finding vertex and intercepts.
Determine the domain and the range of f.
Determine where f is increasing and decreasing.



EXAMPLE

Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

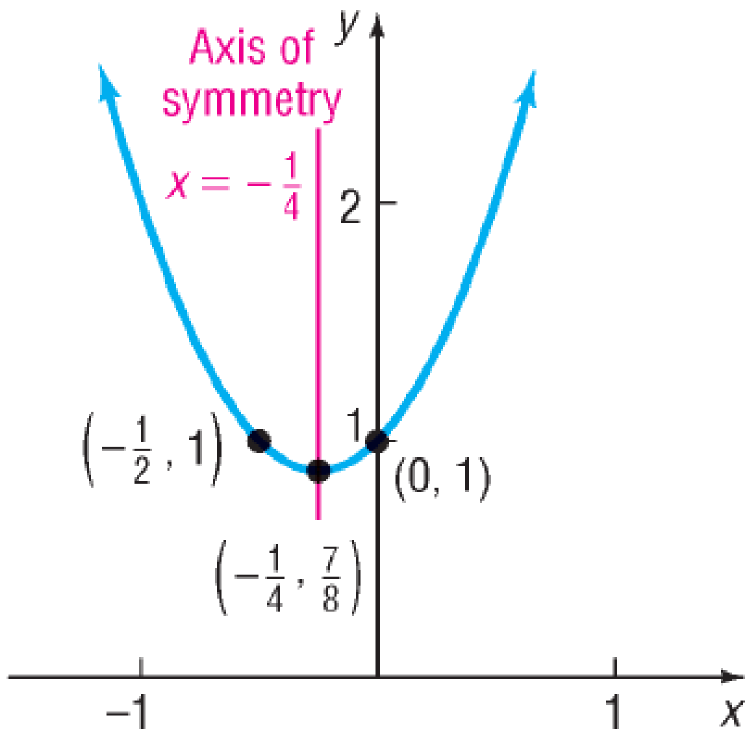
Graph $x^2 - 6x + 9$ by finding vertex and intercepts.
Determine the domain and the range of f.
Determine where f is increasing and decreasing.



EXAMPLE

Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Graph $2x^2 + x + 1$ by finding vertex and intercepts.
Determine the domain and the range of f.
Determine where f is increasing and decreasing.



Given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, we can use

$$f(x) = a(x - h)^2 + k \qquad \textbf{(5)}$$

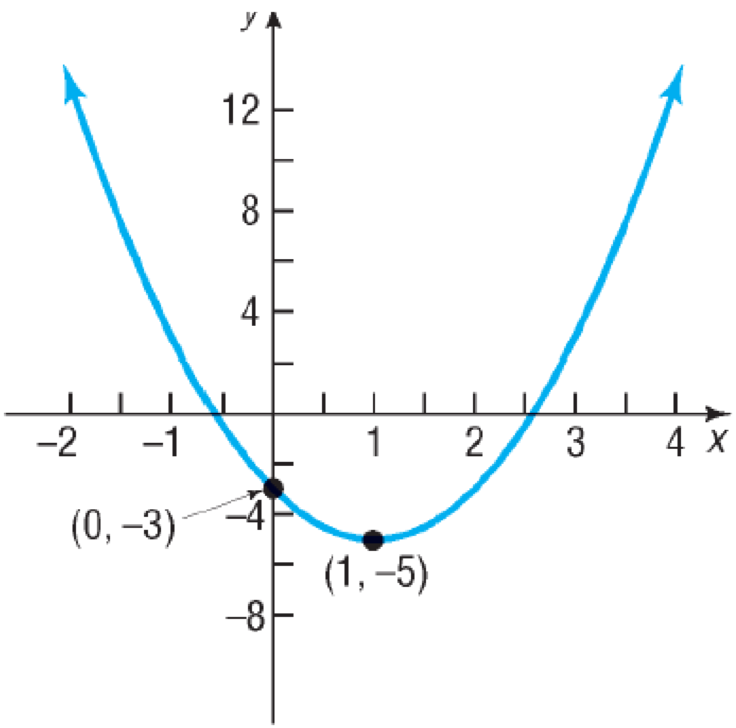
to obtain the quadratic function.

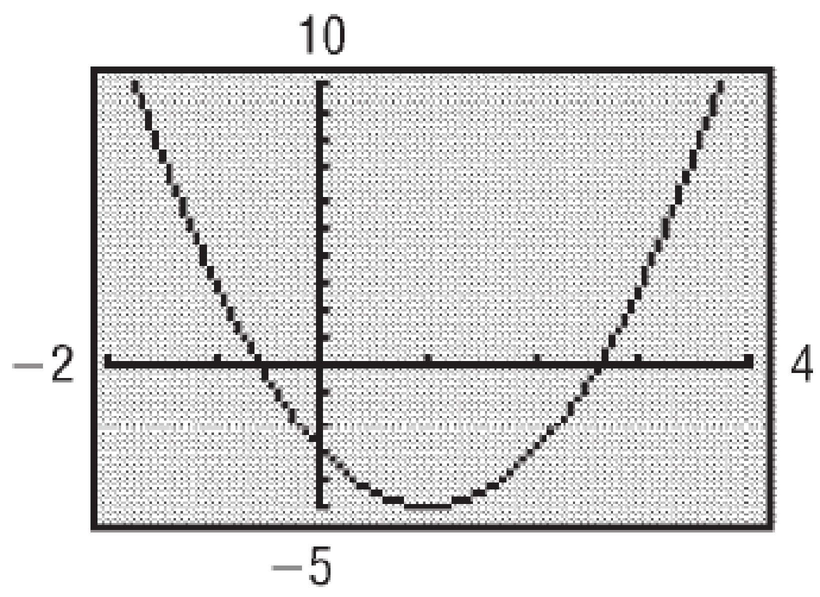
EXAMPLE

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is (1, −5) and whose y-intercept is −3.

$$f(x) = a(x - h)^2 + k = 2(x - 1)^2 - 5 = 2x^2 - 4x - 3$$





Summary

Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$, by Hand.

Option 1

Step 1: Complete the square in x to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

Step 2: Graph the function in stages using transformations.

Option 2

Step 1: Determine the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Step 2: Determine the axis of symmetry, $x = -\frac{b}{2a}$.

Step 3: Determine the y-intercept, $f(0)$.

Step 4: (a) If $b^2 - 4ac > 0$, then the graph of the quadratic function has two x-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.
(b) If $b^2 - 4ac = 0$, the vertex is the x-intercept.
(c) If $b^2 - 4ac < 0$, there are no x-intercepts.

Step 5: Determine an additional point by using the y-intercept and the axis of symmetry.

Step 6: Plot the points and draw the graph.

OBJECTIVE 4

4 **Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems**

EXAMPLE

Finding the Maximum or Minimum Value of a Quadratic Function

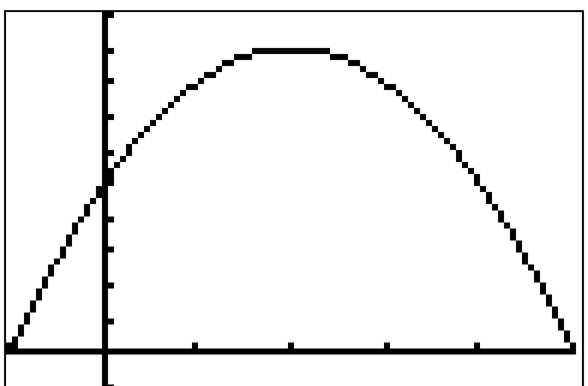
Determine whether the quadratic function

$$f(x) = -x^2 + 4x + 5$$

has a maximum or minimum value.

Then find the maximum or minimum value.

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$



X	Y1	
-1	0	
0	5	
1	8	
2	9	
3	8	
4	5	
5	0	

Y1 = -X² + 4X + 5

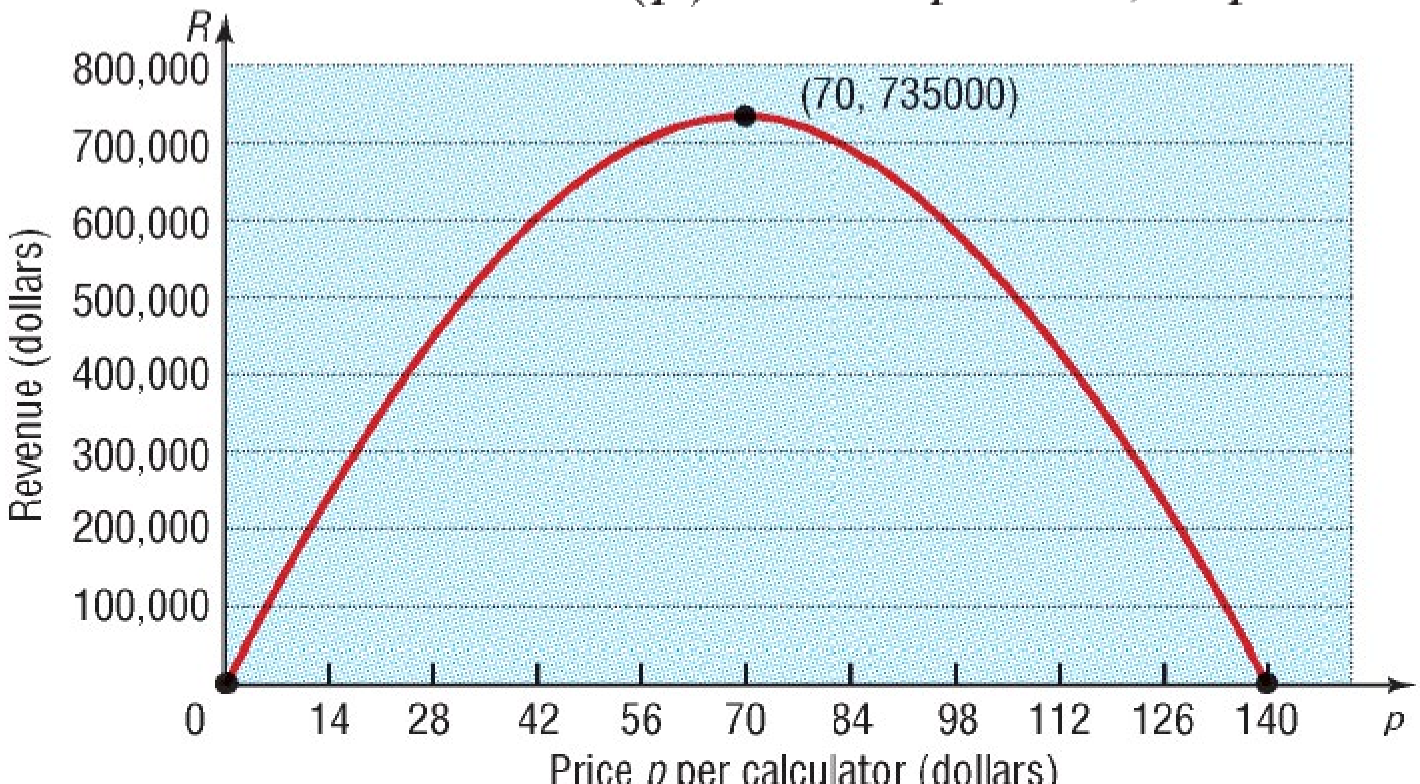
EXAMPLE Maximizing Revenue

The marketing department at Texas Instruments has found that, when certain calculators are sold at a price of p dollars per unit, the revenue R (in dollars) as a function of the price p is

$$R(p) = -150p^2 + 21{,}000p$$

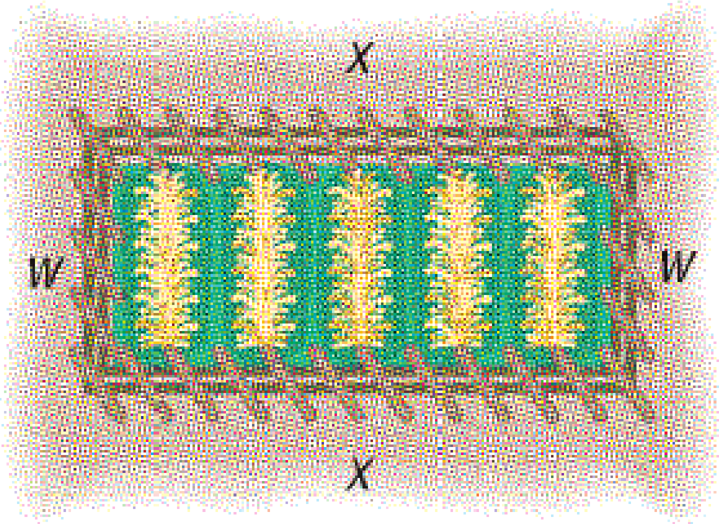
What unit price should be established to maximize revenue? If this price is charged, what is the maximum revenue?

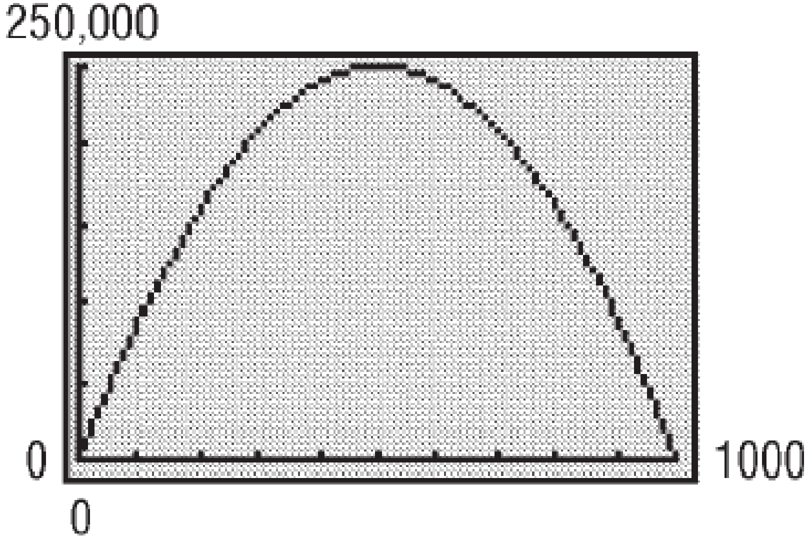
$$R(p) = -150p^2 + 21{,}000p$$



Maximizing the Area Enclosed by a Fence

A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?





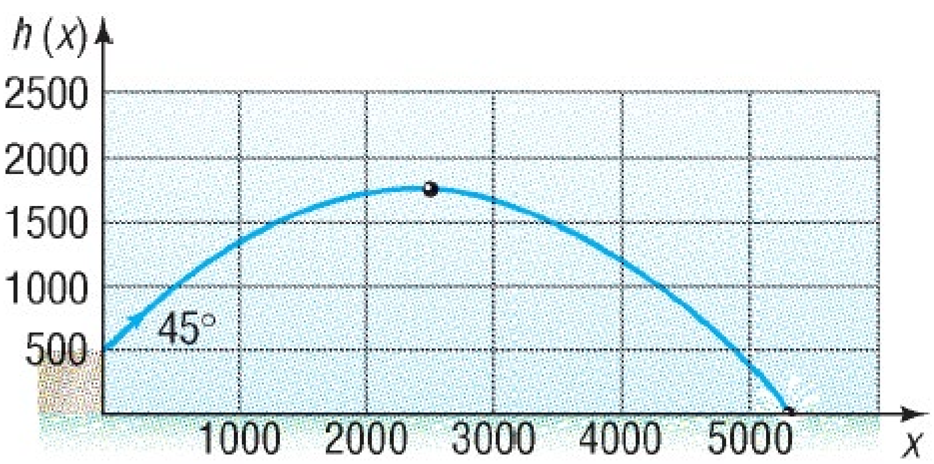
EXAMPLE

Analyzing the Motion of a Projectile

A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. In physics, it is established that the height h of the projectile above the water is given by

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500$$

where x is the horizontal distance of the projectile from the base of the cliff.

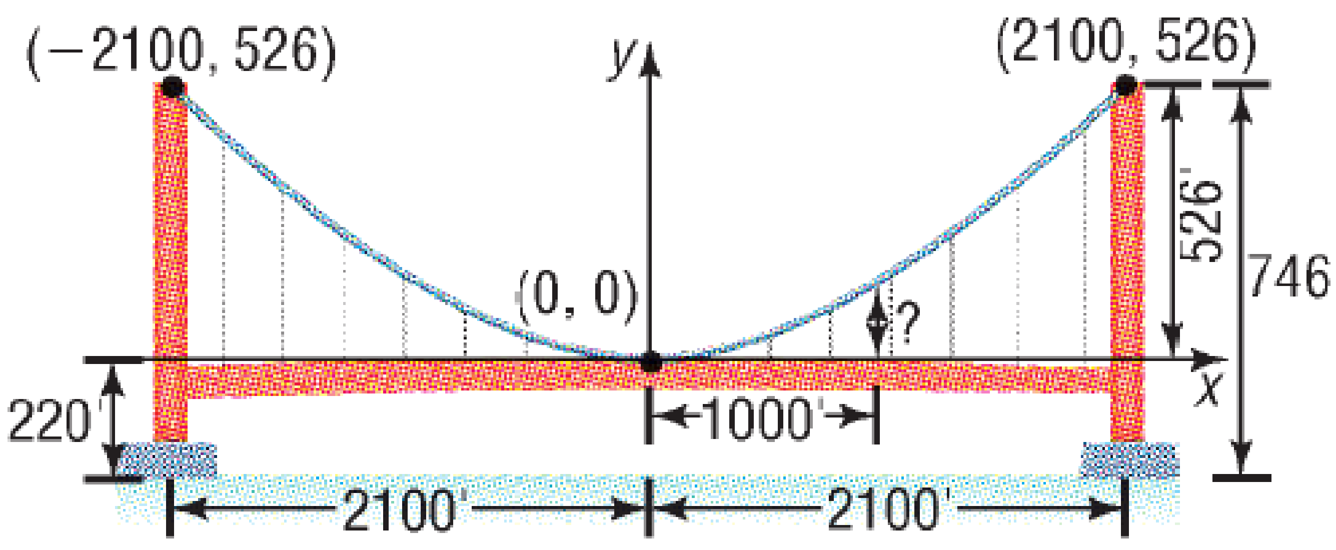


(a) Find the maximum height of the projectile.

(b) How far from the base of the cliff will the projectile strike the water?

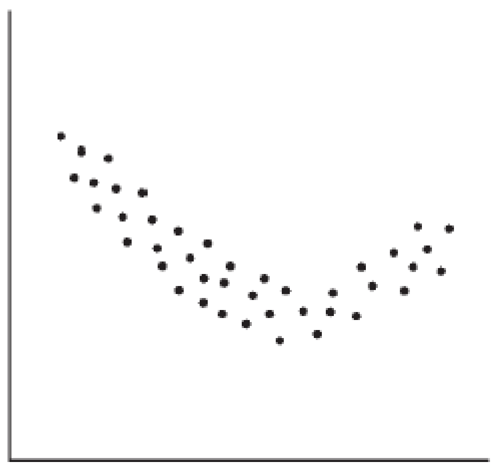
EXAMPLE The Golden Gate Bridge

The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape* and touch the road surface at the center of the bridge. Find the height of the cable at a distance of 1000 feet from the center.

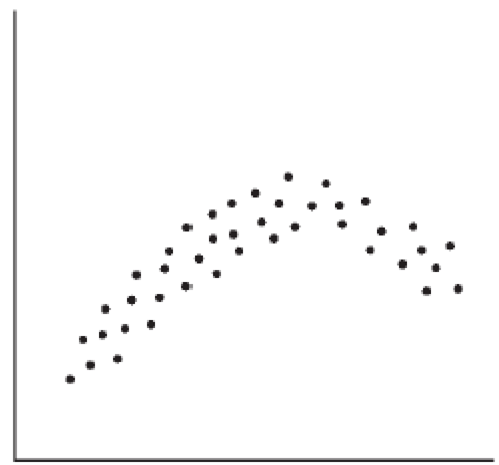


OBJECTIVE 5

5 Use a Graphing Utility to Find the Quadratic Function of Best Fit to Data



$y = ax^2 + bx + c,\ a > 0$



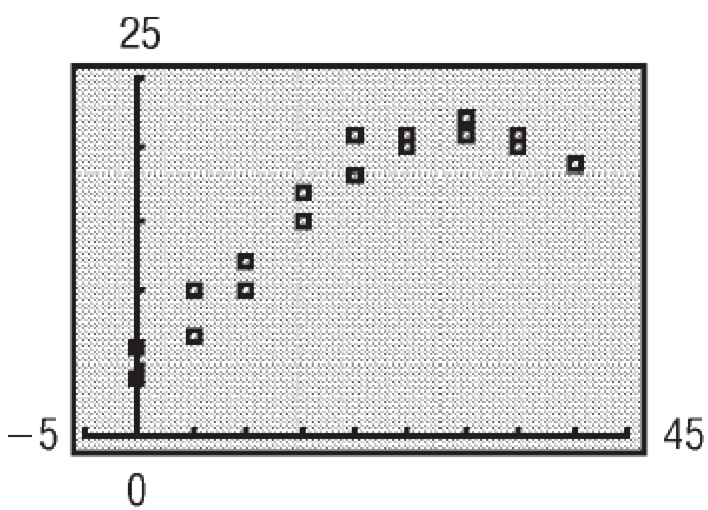
$y = ax^2 + bx + c,\ a < 0$

EXAMPLE

Fitting a Quadratic Function to Data

A farmer collected the data given in Table 3, which shows crop yields Y for various amounts of fertilizer used, x.

(a) With a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.



Plot	Fertilizer, x (Pounds/100 ft^2)	Yield (Bushels)
1	0	4
2	0	6
3	5	10
4	5	7
5	10	12
6	10	10
7	15	15
8	15	17
9	20	18
10	20	21
11	25	20
12	25	21
13	30	21
14	30	22
15	35	21
16	35	20
17	40	19
18	40	19

EXAMPLE

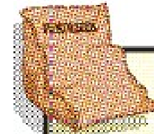
Fitting a Quadratic Function to Data

A farmer collected the data given in Table 3, which shows crop yields Y for various amounts of fertilizer used, x.

(b) Use a graphing utility to find the quadratic function of best fit to these data.

```
QuadReg
 y=ax²+bx+c
 a=⁻.0171212121
 b=1.076515152
 c=3.893939394
```

$$Y(x) = -0.0171x^2 + 1.0765x + 3.8939$$



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EXAMPLE

Fitting a Quadratic Function to Data

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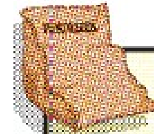

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10	20	21
11	25	20
12	25	21
13	30	21
14	30	22
15	35	21
16	35	20
17	40	19
18	40	19

(c) Use the function found in part (b) to determine the optimal amount of fertilizer to apply.

(d) Use the function found in part (b) to predict crop yield when the optimal amount of fertilizer is applied.

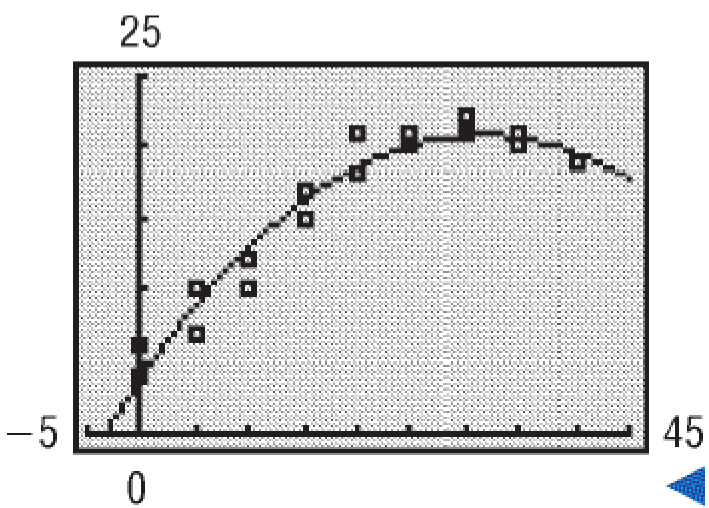
$$Y(x) = -0.0171x^2 + 1.0765x + 3.8939$$

EXAMPLE

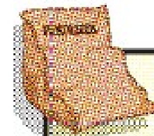
Fitting a Quadratic Function to Data

A farmer collected the data given in Table 3, which shows crop yields Y for various amounts of fertilizer used, x.

(e) Draw the quadratic function of best fit on the scatter diagram.



$$Y(x) = -0.0171x^2 + 1.0765x + 3.8939$$



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10	20	21
11	25	20
12	25	21
13	30	21
14	30	22
15	35	21
16	35	20
17	40	19
18	40	19