

Section 3.2

Polynomial Functions and Models

OBJECTIVE 1

- 1 ✓ Identify Polynomial Functions and Their Degree

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (1)$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain is the set of all real numbers.

EXAMPLE

Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

$$(a) f(x) = 2 - 3x^4$$

$$(b) g(x) = \sqrt{x}$$

$$(c) h(x) = \frac{x^2 - 2}{x^3 - 1}$$

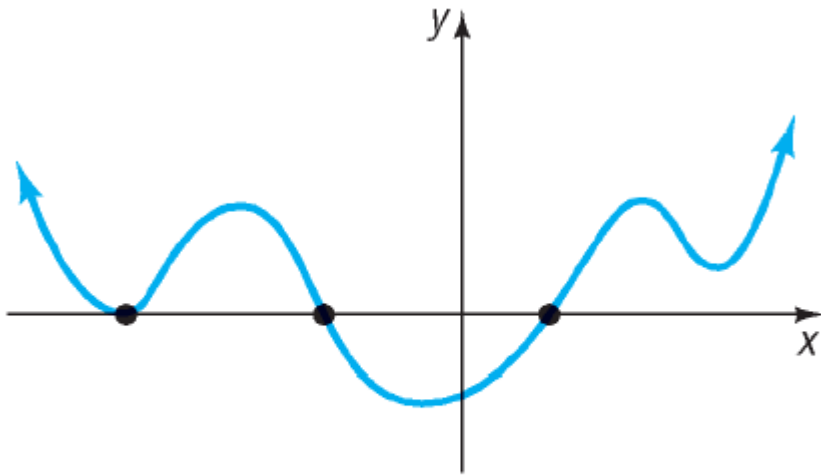
$$(d) F(x) = 0$$

$$(e) G(x) = 8$$

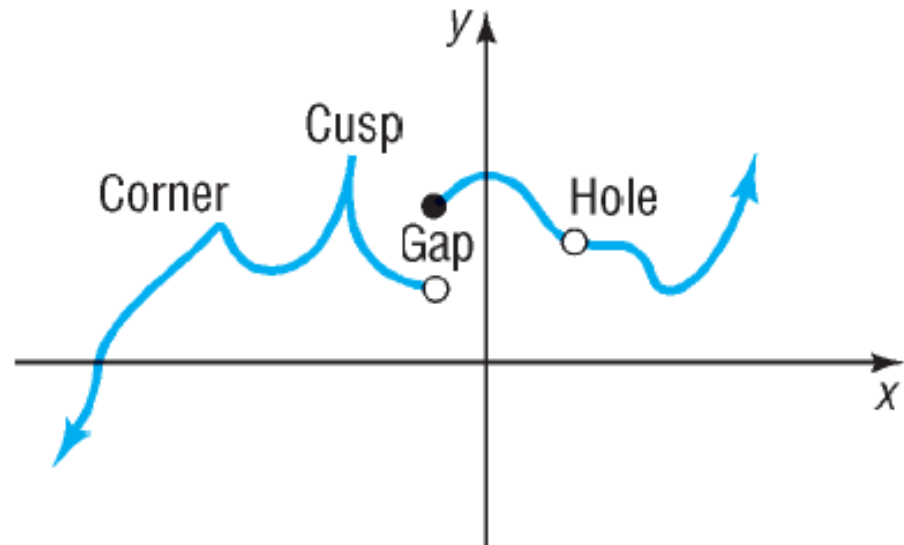
$$(f) H(x) = -2x^3(x - 1)^2$$

Summary of the Properties of the Graphs of Polynomial Functions

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, a_2 \neq 0$	Quadratic function	Parabola: Graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$



(a) Graph of a polynomial function:
smooth, continuous



(b) Cannot be the graph of a
polynomial function

OBJECTIVE 2

2 Graph Polynomial Functions Using Transformations

A **power function of degree n** is a function of the form

$$f(x) = ax^n$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Examples of power functions are

$$f(x) = 3x$$

degree 1

$$f(x) = -5x^2$$

degree 2

$$f(x) = 8x^3$$

degree 3

$$f(x) = -5x^4$$

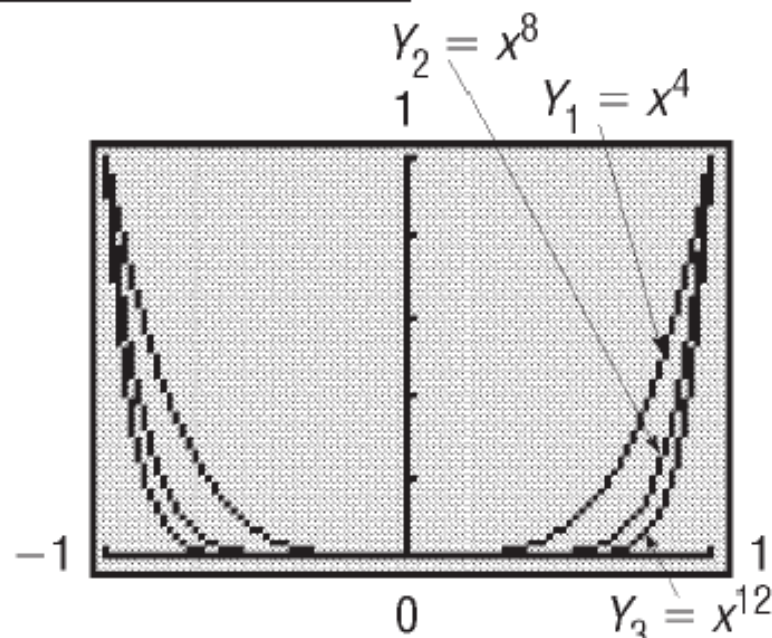
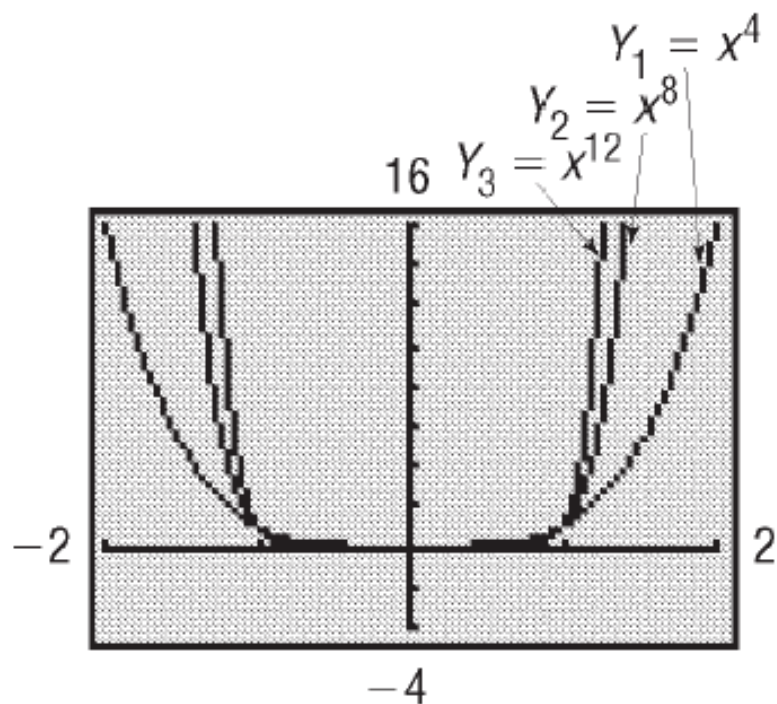
degree 4

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-4 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^4$. On the same screen, graph $Y_2 = g(x) = x^8$. Now, also on the same screen, graph $Y_3 = h(x) = x^{12}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $0 \leq y \leq 1$. What do you notice?

X	Y ₂	Y ₃
-1	1	1
1	1	1
.5	.00391	2.4E-4
.1	1E-8	1E-12
.01	1E-16	1E-24
.001	1E-24	1E-36
0	0	0

Y₂ = X⁸

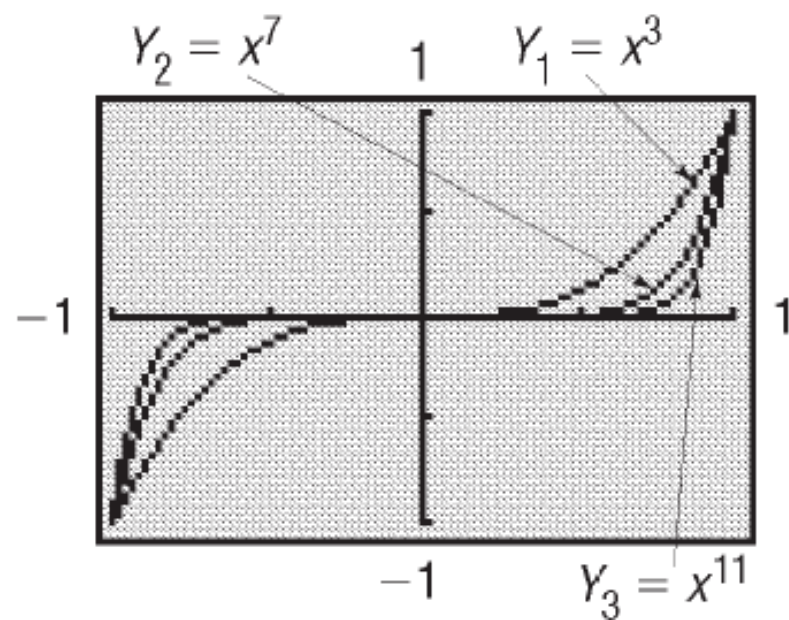
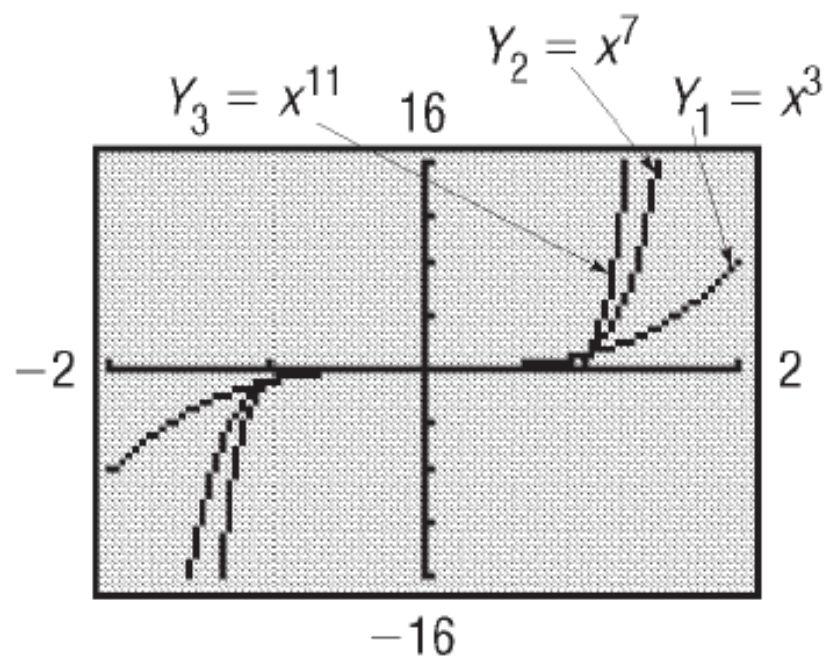


Properties of Power Functions, $f(x) = x^n$, n Is an Even Integer

1. The graph is symmetric with respect to the y -axis, so f is even.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x -axis.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-16 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^3$. On the same screen, graph $Y_2 = g(x) = x^7$ and $Y_3 = h(x) = x^{11}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. What do you notice?



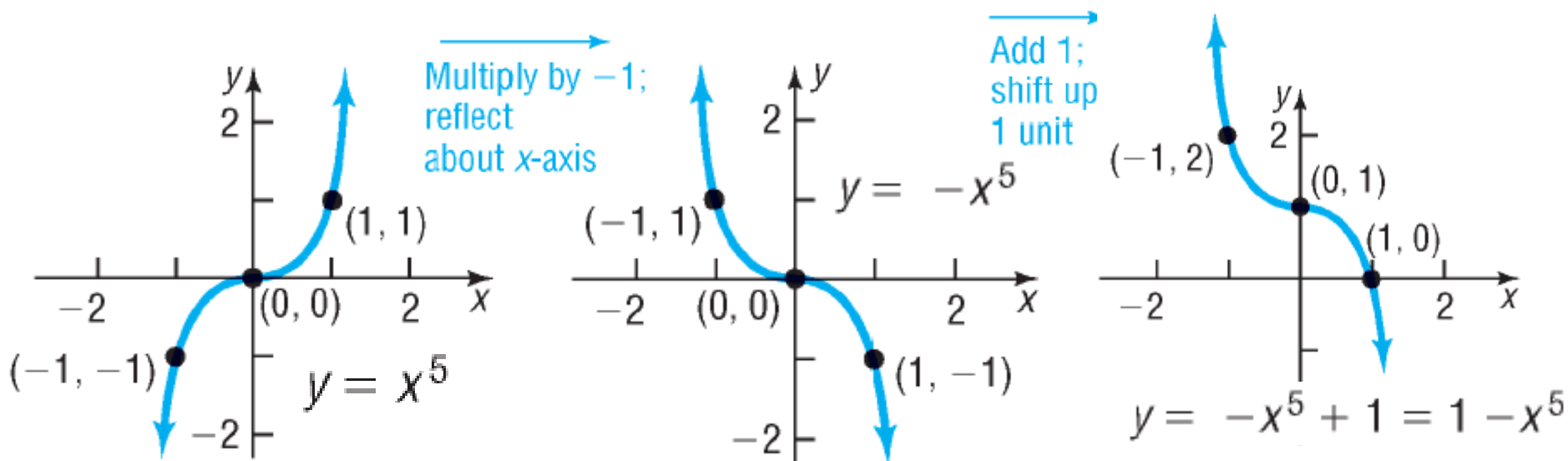
Properties of Power Functions, $f(x) = x^n$, n Is an Odd Integer

- 1.** The graph is symmetric with respect to the origin, so f is odd.
- 2.** The domain and the range are the set of all real numbers.
- 3.** The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
- 4.** As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x -axis.

EXAMPLE

Graphing Polynomial Functions Using Transformations

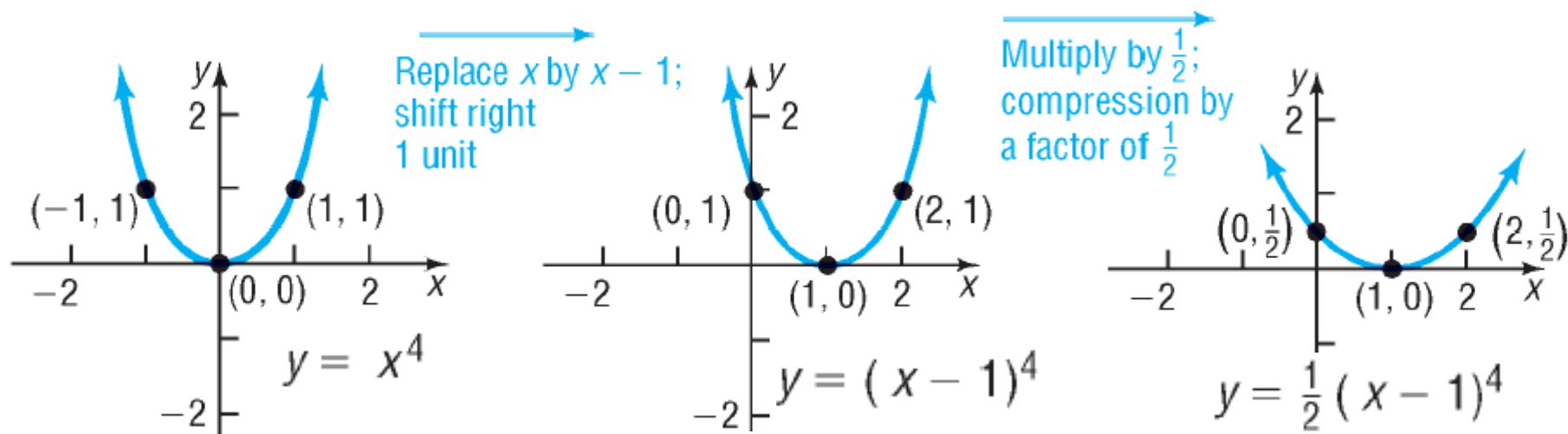
Graph: $f(x) = 1 - x^5$



EXAMPLE

Graphing Polynomial Functions Using Transformations

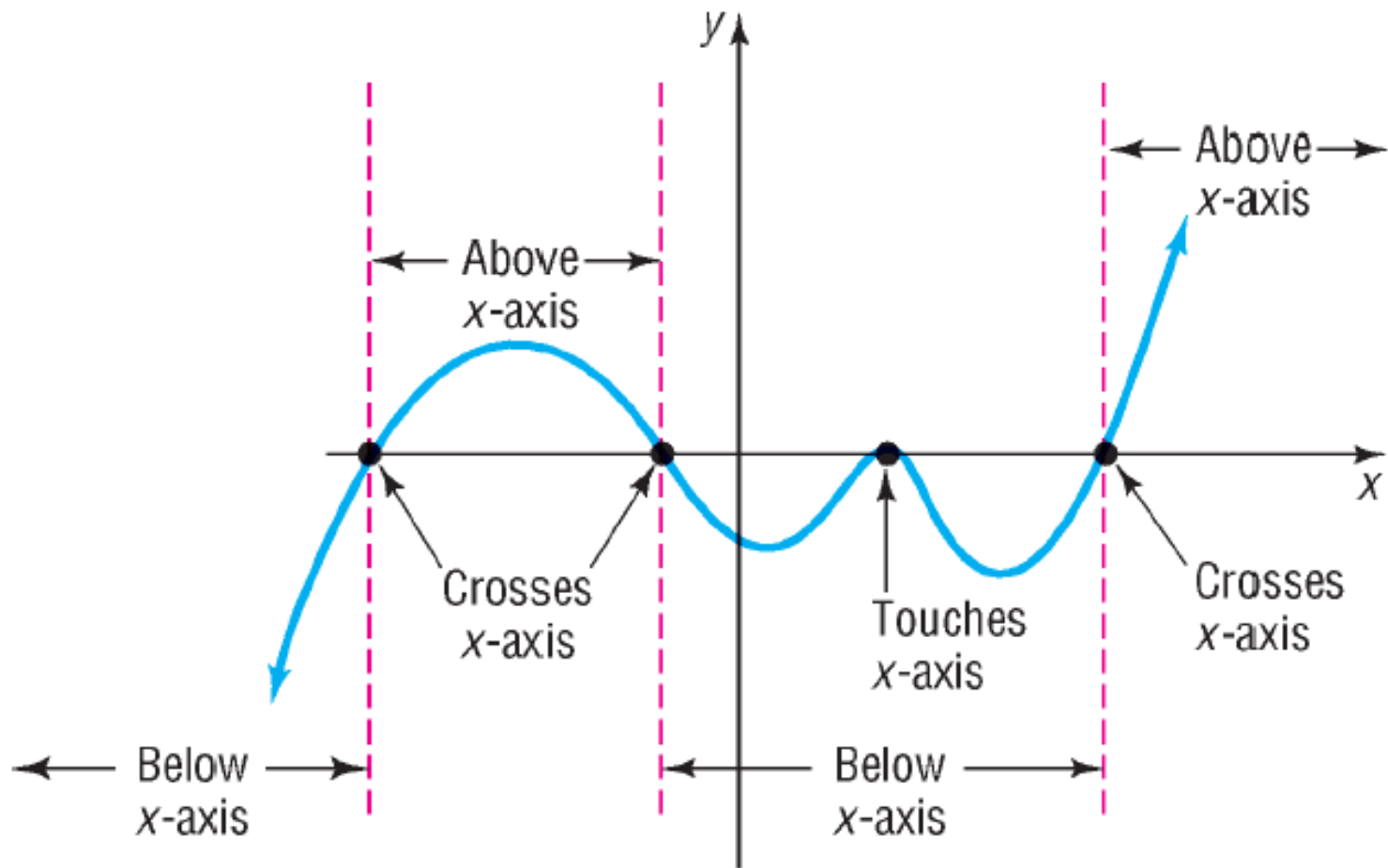
Graph: $f(x) = \frac{1}{2}(x - 1)^4$



OBJECTIVE 3

3 ✓

Identify the Zeros of a Polynomial Function and Their Multiplicity



If f is a polynomial function and r is a real number for which $f(r) = 0$, then r is called a (real) **zero of f** , or **root of f** . If r is a (real) zero of f , then

- (a) r is an x -intercept of the graph of f .
- (b) $(x - r)$ is a factor of f .

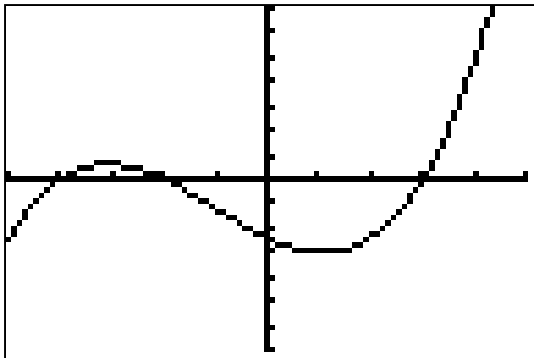
EXAMPLE

Finding a Polynomial from Its Zeros

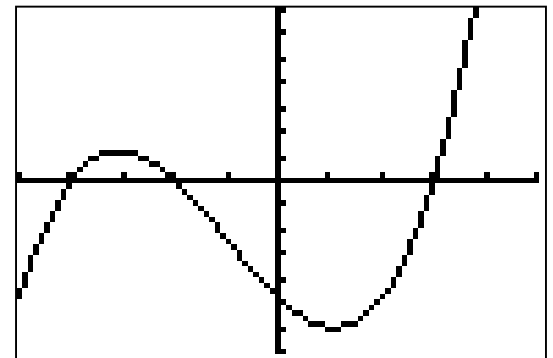
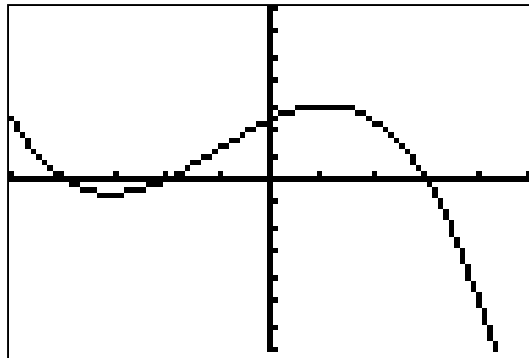
Find a polynomial of degree 3 whose zeros are -4, -2, and 3.

$$f(x) = a(x+4)(x+2)(x-3)$$

$$f(x) = (x+4)(x+2)(x-3)$$



$$f(x) = 2(x+4)(x+2)(x-3)$$



$$f(x) = -(x+4)(x+2)(x-3)$$

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f , then r is called a **zero of multiplicity m of f** .

EXAMPLE

Identifying Zeros and Their Multiplicities

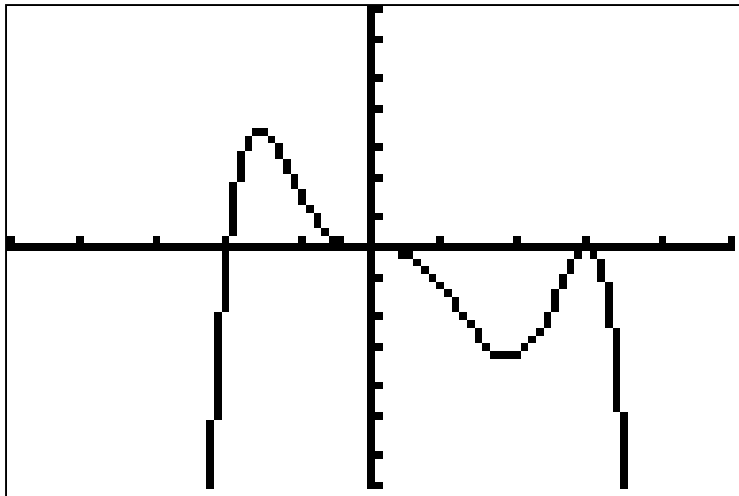
For the polynomial, list all zeros and their multiplicities.

$$f(x) = -2(x - 2)(x + 1)^3(x - 3)^4$$

EXAMPLE

For the polynomial $-x^3(x-3)^2(x+2)$

- Find the x - and y -intercepts of the graph of f .
- Using a graphing utility, graph the polynomial.
- For each x -intercept, determine whether it is of odd or even multiplicity.



X	Y1	
-972		
0		
16		
0		
-12		
-32		
0		

Y1 = -X^3(X-3)^2(X...

X	Y1	
0		
-12		
-32		
0		
-384		
-3500		
-15552		

Y1 = -X^3(X-3)^2(X...

If r Is a Zero of Even Multiplicity

Sign of $f(x)$ does not change from one side of r to the other side of r .

Graph **touches** x -axis at r .

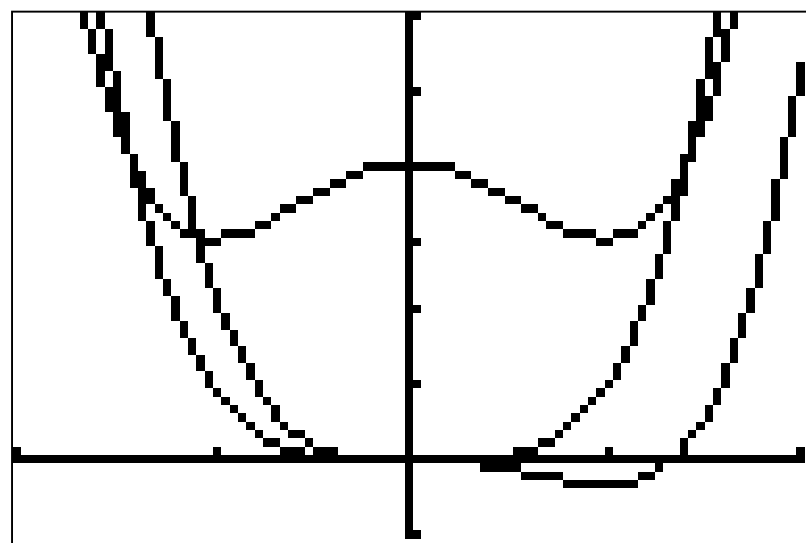
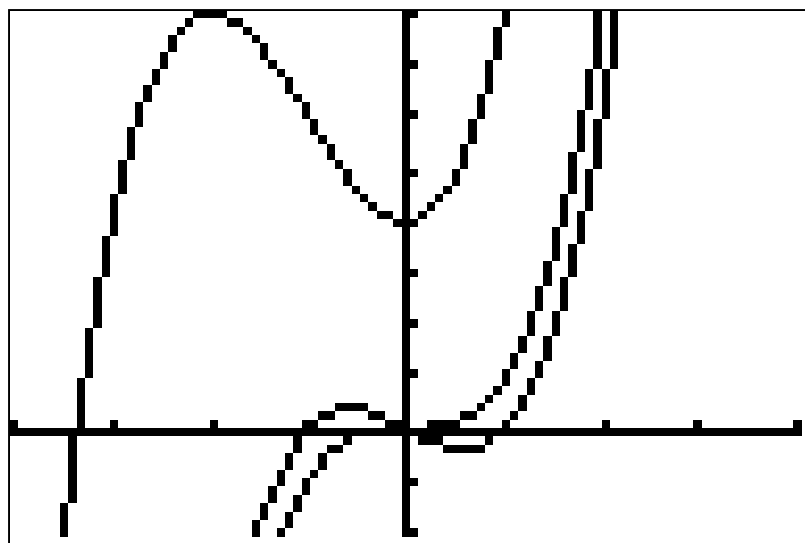
If r Is a Zero of Odd Multiplicity

Sign of $f(x)$ changes from one side of r to the other side of r .

Graph **crosses** x -axis at r .

Exploration

Graph $Y_1 = x^3$, $Y_2 = x^3 - x$, and $Y_3 = x^3 + 3x^2 + 4$. How many turning points do you see? How does the number of turning points relate to the degree? Graph $Y_1 = x^4$, $Y_2 = x^4 - \frac{4}{3}x^3$, and $Y_3 = x^4 - 2x^2$. How many turning points do you see? How does the number of turning points compare to the degree?



Theorem

If f is a polynomial function of degree n , then f has at most $n - 1$ turning points.

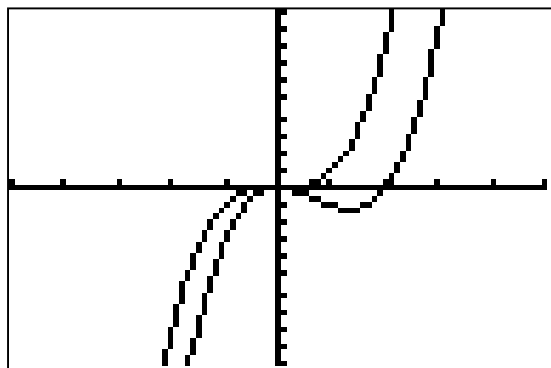
Exploration

For each pair of functions Y_1 and Y_2 given in parts (a), (b), and (c), graph Y_1 and Y_2 on the same viewing window. Create a TABLE or TRACE for large positive and large negative values of x . What do you notice about the graphs of Y_1 and Y_2 as x becomes very large and positive or very large and negative?

(a) $Y_1 = x^2(x - 2)$; $Y_2 = x^3$

(b) $Y_1 = x^4 - 3x^3 + 7x - 3$; $Y_2 = x^4$

(c) $Y_1 = -2x^3 + 4x^2 - 8x + 10$; $Y_2 = -2x^3$



X	Y1	Y2
-6	-288	-216
-5	-175	-125
-4	-96	-64
-3	-54	-27
-2	-16	-8
-1	3	-1
0	0	0

$Y_1 = X^2(X-2)$

X	Y1	Y2
1	1	1
2	8	8
3	27	27
4	64	64
5	125	125
6	216	216
7	343	343

$X=7$

Calculator screens shown for part (a) only.

Theorem

End Behavior

For large values of x , either positive or negative, that is, for large $|x|$, the graph of the polynomial

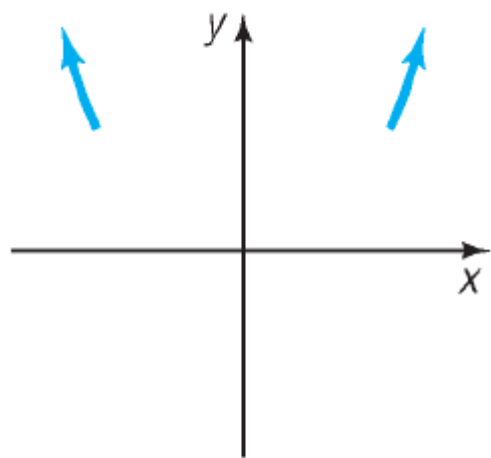
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

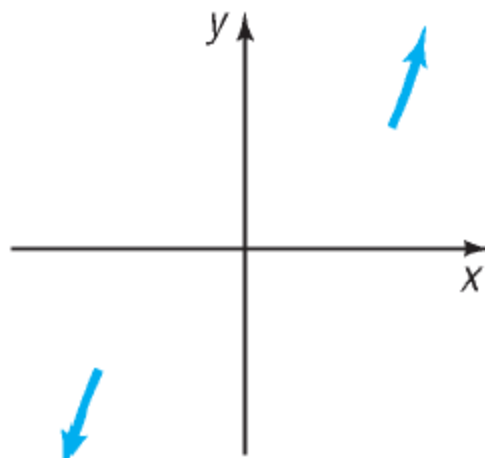
$$y = a_n x^n$$

End Behavior

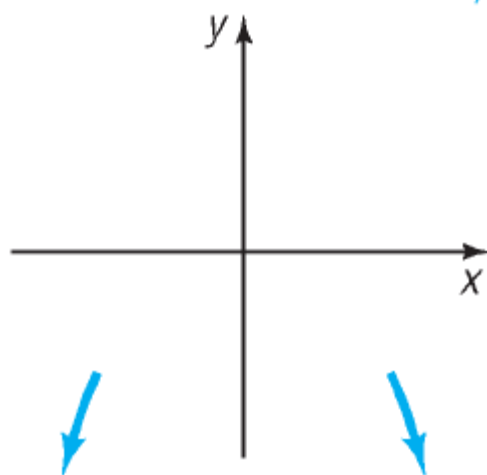
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$



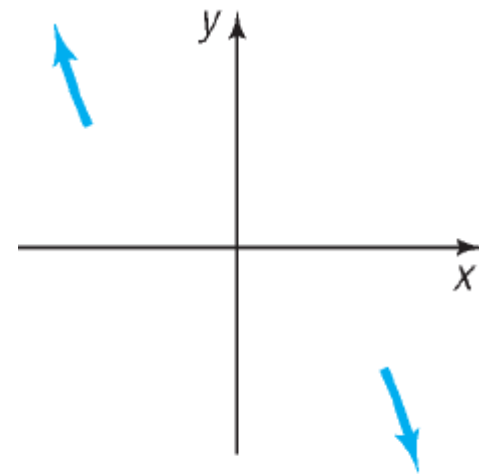
(a)
 $n \geq 2$ even; $a_n > 0$



(c)
 $n \geq 3$ odd; $a_n > 0$



(b)
 $n \geq 2$ even; $a_n < 0$



(d)
 $n \geq 3$ odd; $a_n < 0$

Summary

Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$

Degree of the polynomial f : n

Maximum number of turning points: $n - 1$

At a zero of even multiplicity: The graph of f touches the x -axis.

At a zero of odd multiplicity: The graph of f crosses the x -axis.

Between zeros, the graph of f is either above or below the x -axis.

End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

OBJECTIVE 4

 **Analyze the Graph of a Polynomial Function**

EXAMPLE

Analyzing the Graph of a Polynomial Function

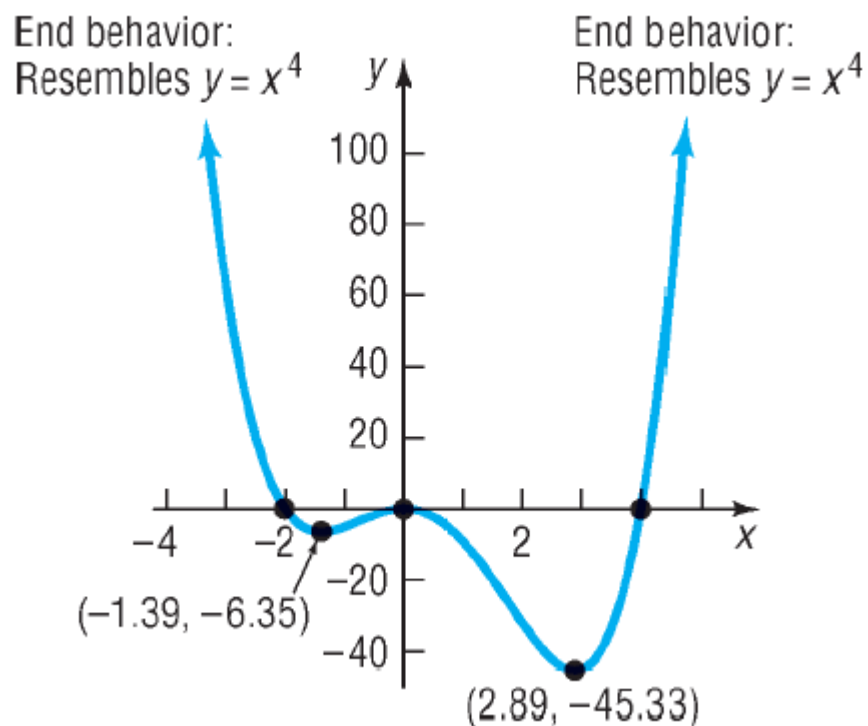
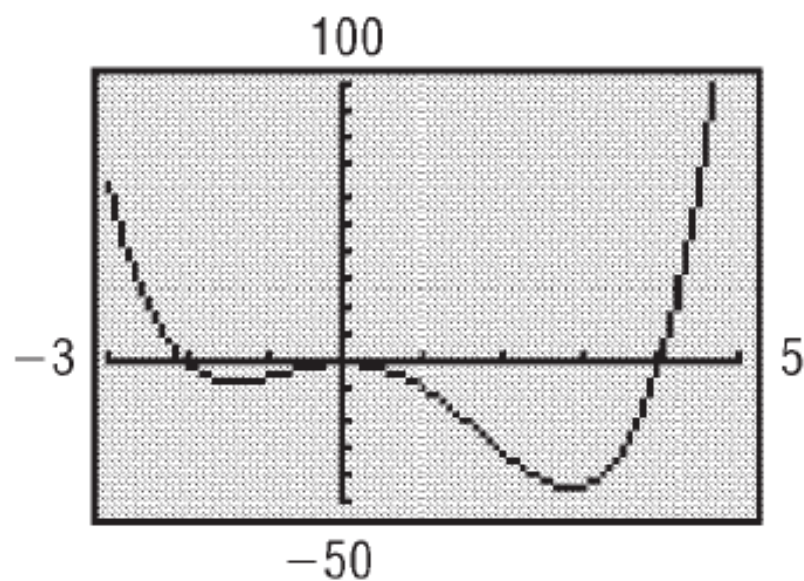
For the polynomial: $f(x) = x^4 - 2x^3 - 8x^2$

- Find the degree of the polynomial. Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.
- Find the x - and y -intercepts of the graph of f .
- Determine whether the graph crosses or touches the x -axis at each x -intercept.
- Use a graphing utility to graph f .
- Determine the number of turning points on the graph of f . Approximate the turning points, if any exist, rounded to two decimal places.
- Use the information obtained in parts (a) to (e) to draw a complete graph of f by hand.
- Find the domain of f . Use the graph to find the range of f .
- Use the graph to determine where f is increasing and where f is decreasing.

EXAMPLE

Analyzing the Graph of a Polynomial Function

For the polynomial: $f(x) = x^4 - 2x^3 - 8x^2$



EXAMPLE

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

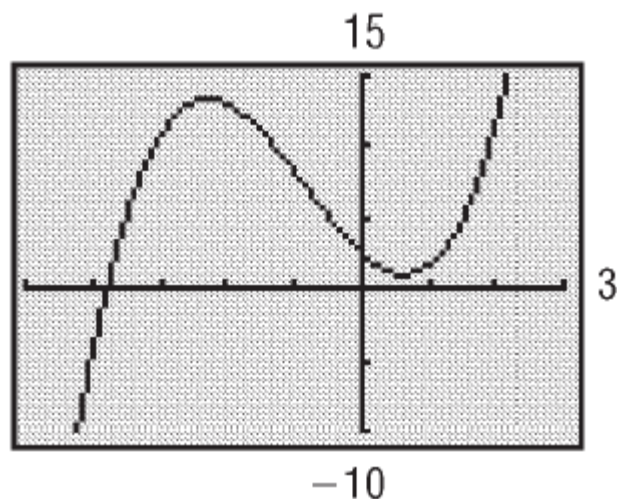
For the polynomial $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$:

- Find the degree of the polynomial. Determine the end behavior: that is, find the power function that the graph of f resembles for large values of $|x|$.
- Graph f using a graphing utility.
- Find the x - and y -intercepts of the graph.
- Use a TABLE to find points on the graph around each x -intercept.
- Determine the local maxima and local minima, if any exist, rounded to two decimal places. That is, locate any turning points.
- Use the information obtained in parts (a)–(e) to draw a complete graph of f by hand. Be sure to label the intercepts, turning points, and the points obtained in part (d).
- Find the domain of f . Use the graph to find the range of f .
- Use the graph to determine where f is increasing and where f is decreasing.

EXAMPLE

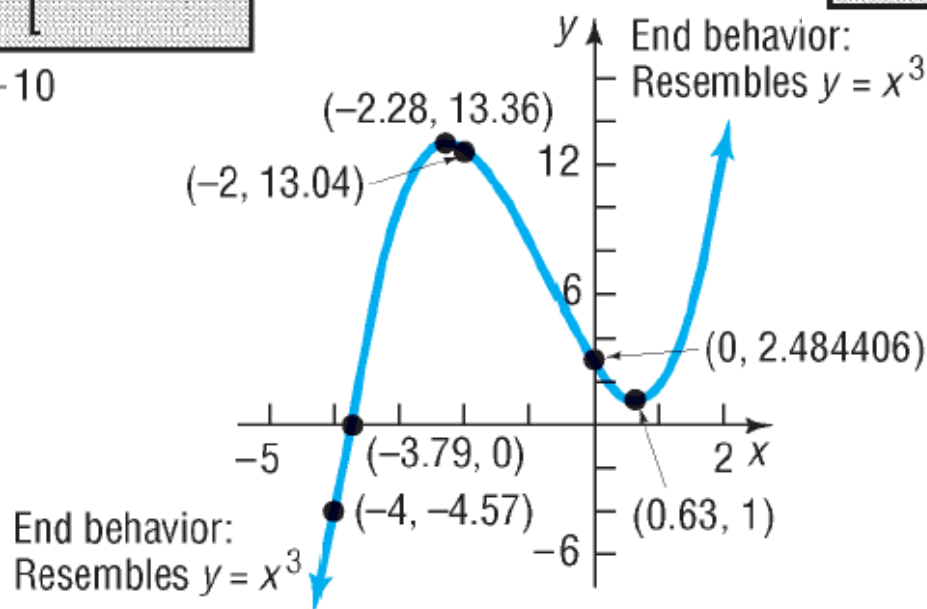
Using a Graphing Utility to Analyze the Graph of a Polynomial Function

For the polynomial $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$:



X	Y1
-4	-4.574
-2	13.035

Y1 = X^3 + 2.48X^2 - 4...



Summary

Steps for Graphing a Polynomial by Hand

To analyze the graph of a polynomial function $y = f(x)$, follow these steps:

STEP 1: End behavior: find the power function that the graph of f resembles for large values of x .

STEP 2: (a) Find the x -intercepts, if any, by solving the equation $f(x) = 0$.

(b) Find the y -intercept by letting $x = 0$ and finding the value of $f(0)$.

STEP 3: Determine whether the graph of f crosses or touches the x -axis at each x -intercept.

STEP 4: Use a graphing utility to graph f . Determine the number of turning points on the graph of f . Approximate any turning points rounded to two decimal places.

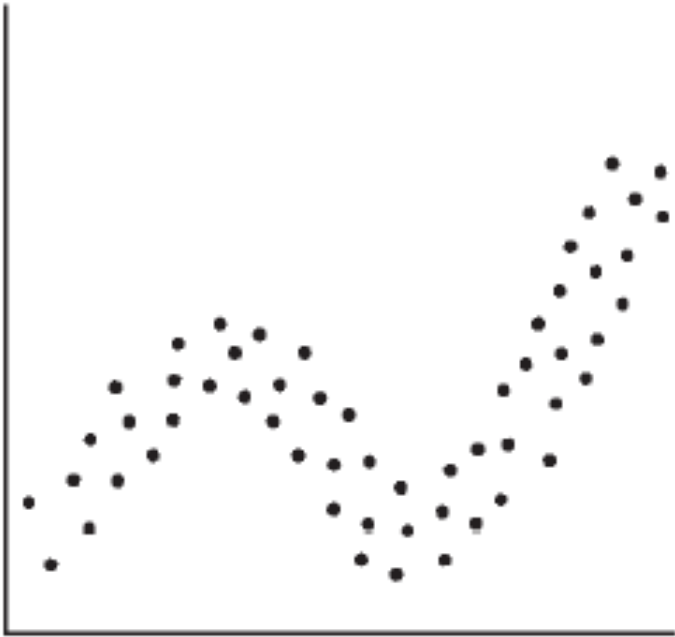
STEP 5: Use the information obtained in Steps 1 to 4 to draw a complete graph of f by hand.

OBJECTIVE 5

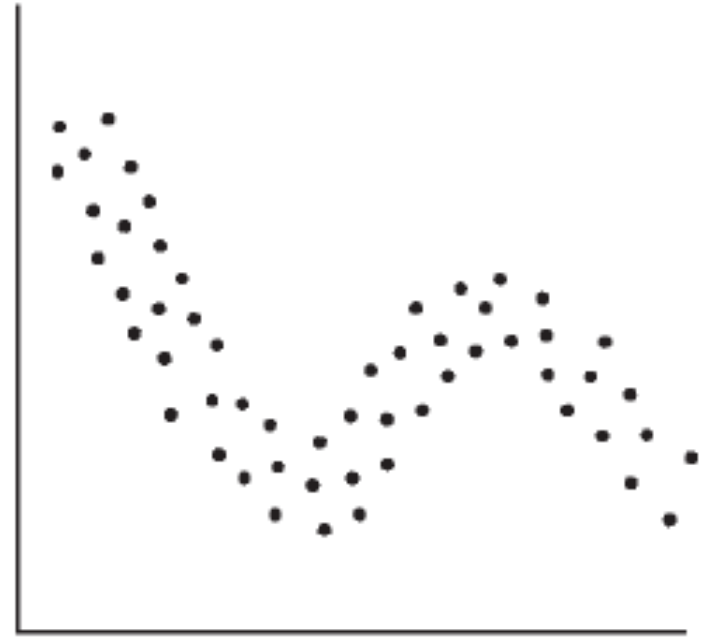


Find the Cubic Function of Best Fit to Data

Data That Follow A Cubic Relation



$$y = ax^3 + bx^2 + cx + d, a > 0$$



$$y = ax^3 + bx^2 + cx + d, a < 0$$

EXAMPLE

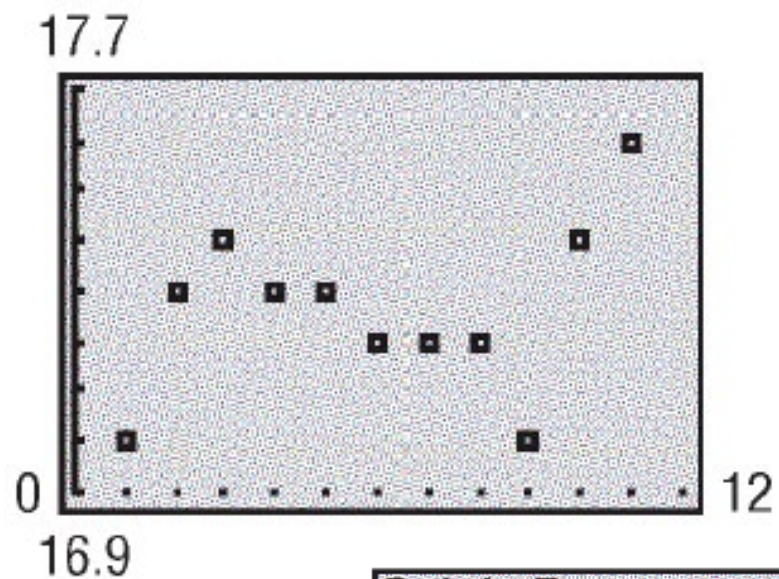
A Cubic Function of Best Fit

The data in Table 8 represent the average miles per gallon of vans, pickups, and sport utility vehicles (SUVs) in the United States for 1991–2001, where 1 represents 1991, 2 represents 1992, and so on.

- Draw a scatter diagram of the data using the year x as the independent variable and average miles per gallon M as the dependent variable. Comment on the type of relation that may exist between the two variables x and M .
- Using a graphing utility, find the cubic function of best fit $M = M(x)$ to these data.
- Graph the cubic function of best fit on your scatter diagram.
- Use the function found in part (b) to predict the average miles per gallon in 2002 ($x = 12$).

Table on next slide.

Year, x	Average Miles per Gallon, M
1991, 1	17.0
1992, 2	17.3
1993, 3	17.4
1994, 4	17.3
1995, 5	17.3
1996, 6	17.2
1997, 7	17.2
1998, 8	17.2
1999, 9	17.0
2000, 10	17.4
2001, 11	17.6



CubicReg
 $y = ax^3 + bx^2 + cx + d$
 $a = .0058080808$
 $b = -.1006993007$
 $c = .49500777$
 $d = 16.62575758$

