Section 3.3

Properties of Rational Functions

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial.

The domain is the set of all real numbers except those for which the denominator q is 0.

OBJECTIVE 1

1 Find the Domain of a Rational Function

EXAMPLE Finding the Domain of a Rational Function

(a) The domain of $R(x) = \frac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, $\{x | x \neq -5\}$.

(b) The domain of $R(x) = \frac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2, that is, $\{x | x \neq -2, x \neq 2\}$.

(c) The domain of $R(x) = \frac{x^3}{x^2 + 1}$ is the set of all real numbers.

(d) The domain of $R(x) = \frac{-x^2 + 2}{3}$ is the set of all real numbers.

(e) The domain of $R(x) = \frac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1, that is, $\{x | x \neq 1\}$.

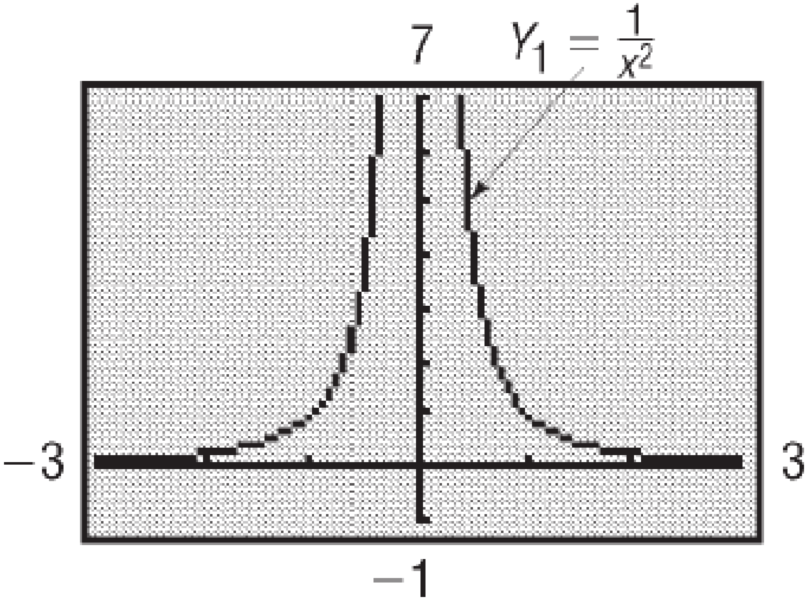
It is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of R is $\{x | x \neq 1\}$ and the domain of f is the set of all real numbers.

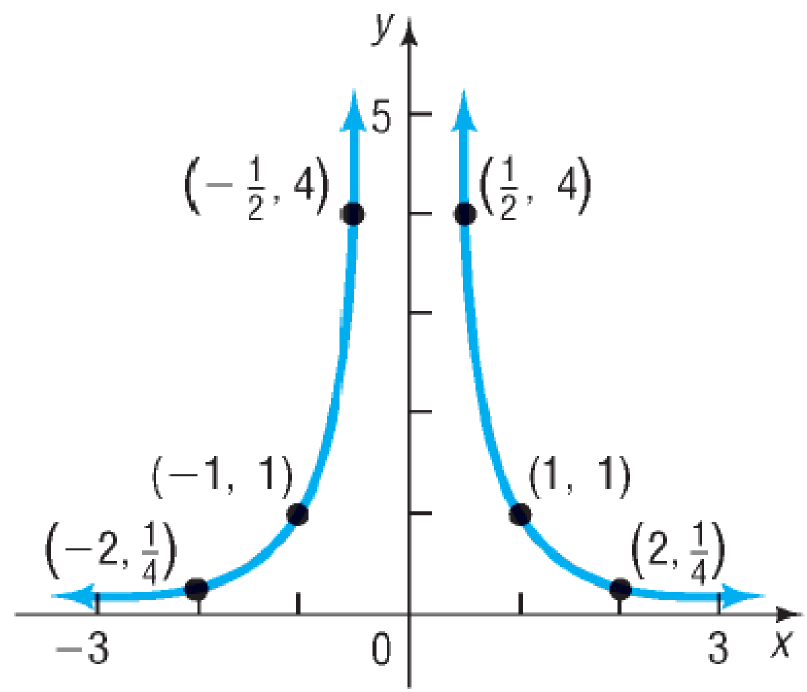
EXAMPLE Graphing $y = \dfrac{1}{x^2}$

Analyze the graph of $H(x) = \dfrac{1}{x^2}$.



X	Y1
.1	100
.01	10000
.001	1E6
1E−4	1E8
10	.01
100	1E−4
1000	1E−6

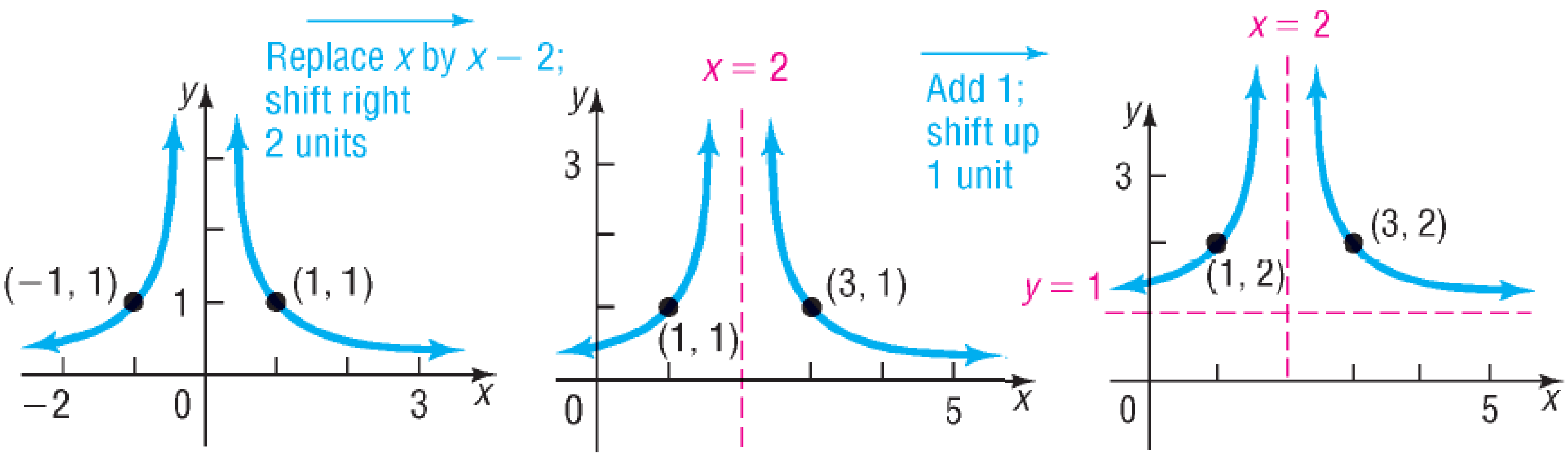
Y1=1/X²



EXAMPLE

Using Transformations to Graph a Rational Function

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$



$y = \dfrac{1}{x^2}$ $y = \dfrac{1}{(x-2)^2}$ $y = \dfrac{1}{(x-2)^2} + 1$

Asymptotes

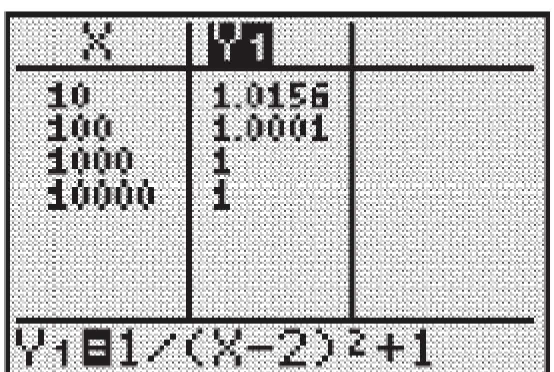
Exploration

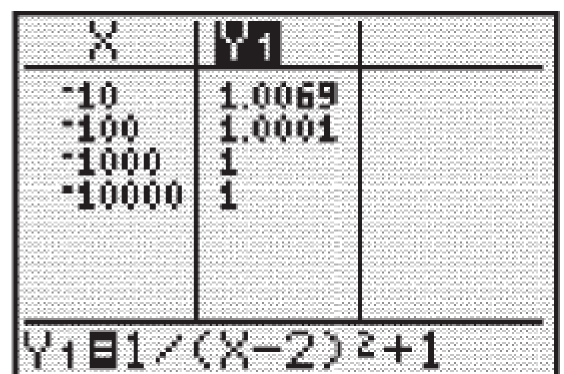
(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \frac{1}{(x-2)^2} + 1$ at $x = 10, 100, 1000$, and $10{,}000$. What happens to the values of H as x becomes unbounded in the positive direction, symbolized by $\lim_{x\to\infty} H(x)$?

(b) Evaluate H at $x = -10, -100, -1000$, and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, symbolized by $\lim_{x\to-\infty} H(x)$?

(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999$, and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, symbolized by $\lim_{x\to 2^-} H(x)$?

(d) Evaluate H at $x = 2.5, 2.1, 2.01, 2.001$, and 2.0001. What happens to the values of H as x approaches 2, $x > 2$, symbolized by $\lim_{x\to 2^+} H(x)$?





Asymptotes

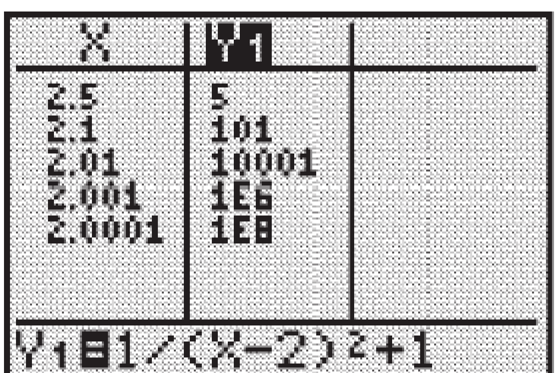
Exploration

(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \dfrac{1}{(x-2)^2} + 1$ at $x = 10, 100, 1000,$ and $10{,}000$. What happens to the values of H as x becomes unbounded in the positive direction, symbolized by $\lim_{x \to \infty} H(x)$?

(b) Evaluate H at $x = -10, -100, -1000,$ and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, symbolized by $\lim_{x \to -\infty} H(x)$?

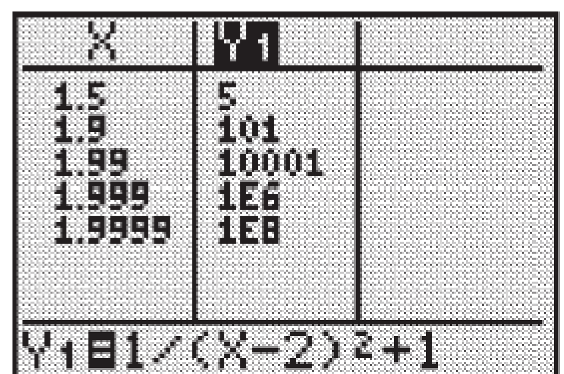
(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999,$ and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, symbolized by $\lim_{x \to 2^-} H(x)$?

(d) Evaluate H at $x = 2.5, 2.1, 2.01, 2.001,$ and 2.0001. What happens to the values of H as x approaches 2, $x > 2$, symbolized by $\lim_{x \to 2^+} H(x)$?



X	Y1
2.5	5
2.1	101
2.01	10001
2.001	1E6
2.0001	1E8

Y1=1/(X−2)²+1



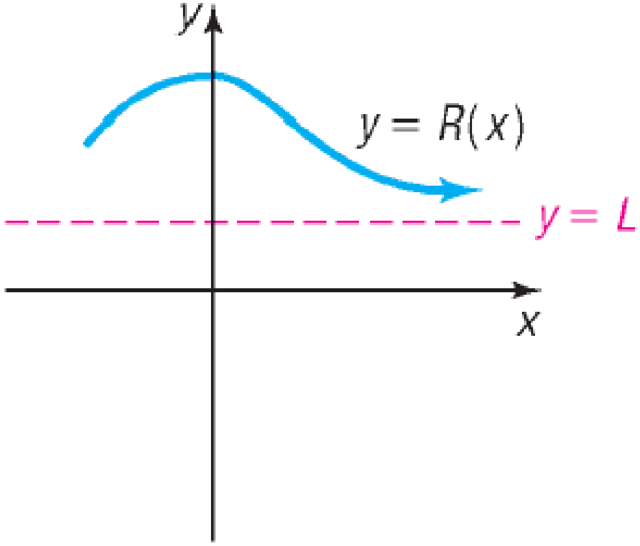
X	Y1
1.5	5
1.9	101
1.99	10001
1.999	1E6
1.9999	1E8

Y1=1/(X−2)²+1

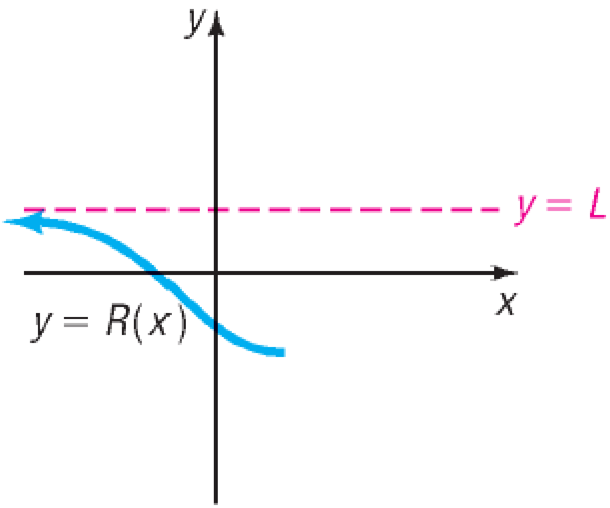
Let R denote a function:

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.

If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.



(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

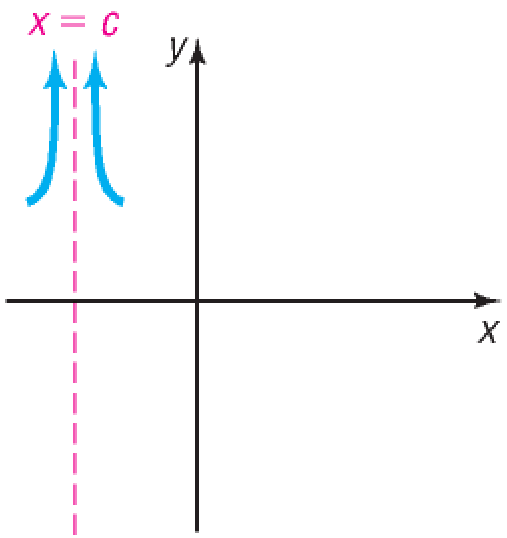


(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x\to -\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

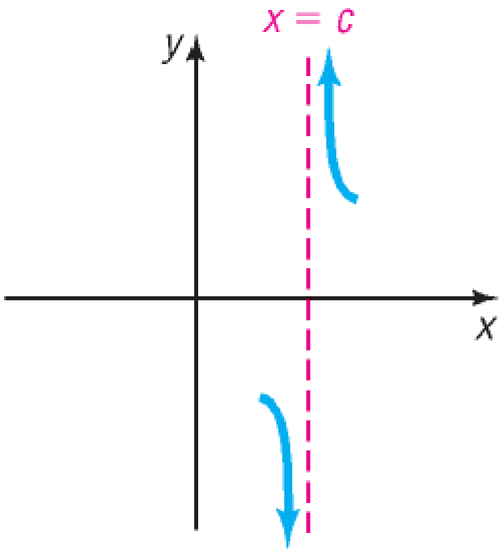
Let R denote a function:

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.

If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

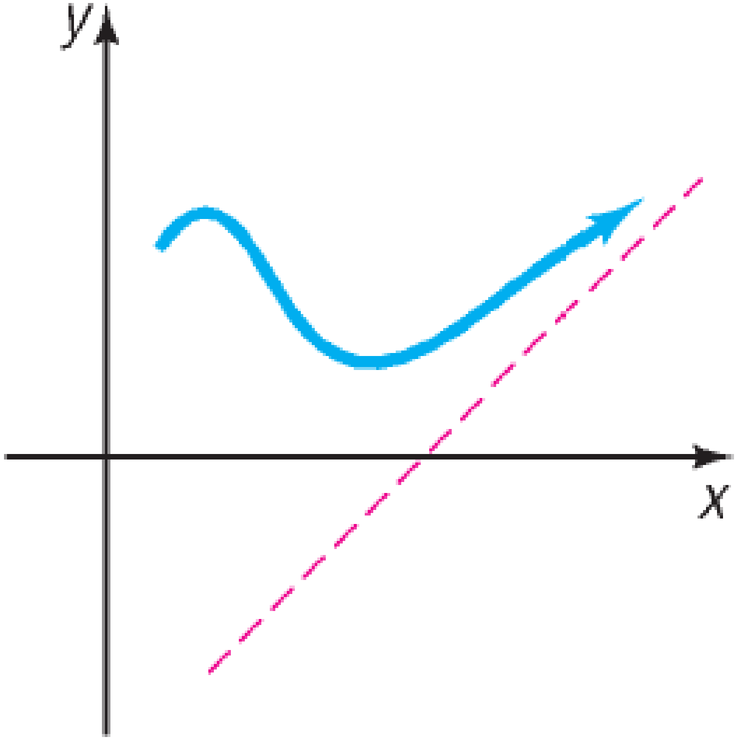


(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x \to c^-} R(x) = \infty$; $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x \to c^-} R(x) = -\infty$; $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Oblique asymptote



OBJECTIVE 2

2 Find the Vertical Asymptotes of a Rational Function

Theorem

Locating Vertical Asymptotes

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, then R will have the vertical asymptote $x = r$.

EXAMPLE Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

$$R(x)=\frac{5x^2}{3+x}$$

$x=-3$

$$H(x)=\frac{x-3}{(x+2)(x-2)}$$

$x=2$ and $x=-2$

$$F(x)=\frac{x-1}{x^2+5x+4}$$

$x=-4$ and $x=-1$

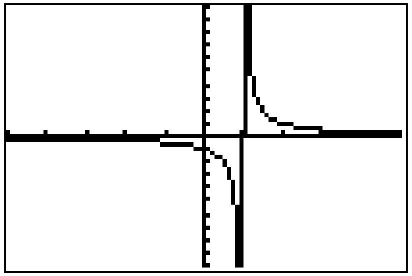
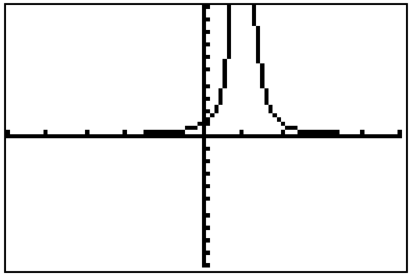
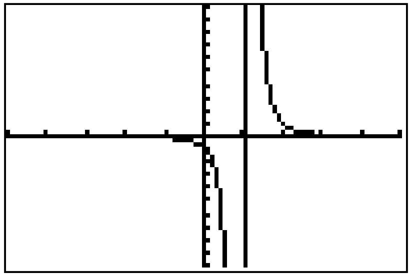
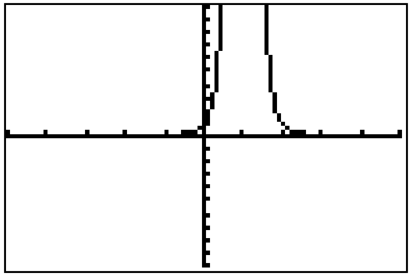
$$G(x)=\frac{x^2+3x+2}{x^2-4}$$

$x=2$

Exploration

Graph each of the following rational functions:

$$R(x) = \frac{1}{x - 1} \quad R(x) = \frac{1}{(x - 1)^2} \quad R(x) = \frac{1}{(x - 1)^3} \quad R(x) = \frac{1}{(x - 1)^4}$$

Each has the vertical asymptote $x = 1$. What happens to the value of $R(x)$ as x approaches 1 from the right side of the vertical asymptote; that is, what is $\lim_{x \to 1^+} R(x)$? What happens to the value of $R(x)$ as x approaches 1 from the left side of the vertical asymptote; that is, what is $\lim_{x \to 1^-} R(x)$? How does the multiplicity of the zero in the denominator affect the graph of R?

OBJECTIVE 3

3 Find the Horizontal or Oblique Asymptotes of a Rational Function

Theorem

If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

EXAMPLE Finding Horizontal Asymptotes

Find the horizontal asymptotes, if any, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

X	Y1	
-10	-.0563	
-100	-.0028	
-1000	-3E-4	
-10000	-3E-5	
-1E5	-3E-6	
-1E6	-3E-7	
-1E7	-3E-8	

$Y_1 = (X-12)/(4X^2+\ldots$

X	Y1	
10	-.0049	
100	.00219	
1000	2.5E-4	
10000	2.5E-5	
100000	2.5E-6	
1E6	2.5E-7	
1E7	2.5E-8	

$Y_1 = (X-12)/(4X^2+\ldots$

If a rational function $R(x) = \frac{p(x)}{q(x)}$ is **improper**,

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

1. If $f(x) = b$, a constant, then the line $y = b$ is a horizontal asymptote of the graph of R.
2. If $f(x) = ax + b$, $a \neq 0$, then the line $y = ax + b$ is an oblique asymptote of the graph of R.
3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

X	Y1	Y2
-10	-27.21	-27
-100	-297	-297
-1000	-2997	-2997
-10000	-29997	-29997
-1E5	-3E5	-3E5
-1E6	-3E6	-3E6
-1E7	-3E7	-3E7

Y1 = (3X^4-X^2)/(X...

X	Y1	Y2
10	33.185	33
100	303.02	303
1000	3003	3003
10000	30003	30003
100000	300003	300003
1E6	3E6	3E6
1E7	3E7	3E7

Y1 = (3X^4-X^2)/(X...

If $f(x) = ax + b$, $a \neq 0$, then the line $y = ax + b$ is an oblique asymptote of the graph of R.

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

If $f(x) = b$, a constant, then the line $y = b$ is a horizontal asymptote of the graph of R.

EXAMPLE

Finding Horizontal or Oblique Asymptotes

Find the horizontal or oblique asymptotes, if any, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Summary

Finding Horizontal and Oblique Asymptotes of a Rational Function R

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$, then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$, then R is improper. Here long division is used.
   (a) If $n = m$, the quotient obtained will be a number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$, the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n > m + 1$, the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of R will behave like the graph of the quotient.

NOTE The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■