Section 3.4

The Graph of a Rational Function;

Inverse and Joint Variation

OBJECTIVE 1

1 Analyze the Graph of a Rational Function

Analyzing the Graph of a Rational Function

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. The x-intercepts, if any, of $R(x) = \frac{p(x)}{q(x)}$ in lowest terms, satisfy the equation $p(x) = 0$. The y-intercept, if there is one, is $R(0)$.

STEP 4: Test for symmetry. Replace x by $-x$ in $R(x)$. If $R(-x) = R(x)$, there is symmetry with respect to the y-axis; if $R(-x) = -R(x)$, there is symmetry with respect to the origin.

STEP 5: Locate the vertical asymptotes. The vertical asymptotes, if any, of $R(x) = \frac{p(x)}{q(x)}$ in lowest terms are found by identifying the real zeros of $q(x)$. Each zero of the denominator gives rise to a vertical asymptote.

STEP 6: Locate the horizontal or oblique asymptotes, if any, using the procedure given in Section 3.3. Determine points, if any, at which the graph of R intersects these asymptotes.

STEP 7: Graph R using a graphing utility.

STEP 8: Use the results obtained in Steps 1 through 7 to graph R by hand.

EXAMPLE

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

$$R(x) = \frac{x-1}{(x+2)(x-2)}$$

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph.

x-intercepts $p(x) = 0$. y-intercept $R(0)$

STEP 4: Test for symmetry.

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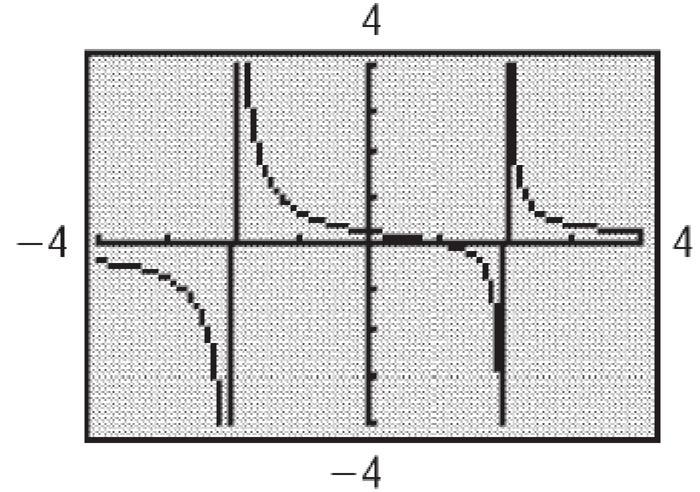

Step 5: Locate the vertical asymptotes.

Step 6: Locate the horizontal or oblique asymptotes, if any.

Step 7: Graph R using a graphing utility.

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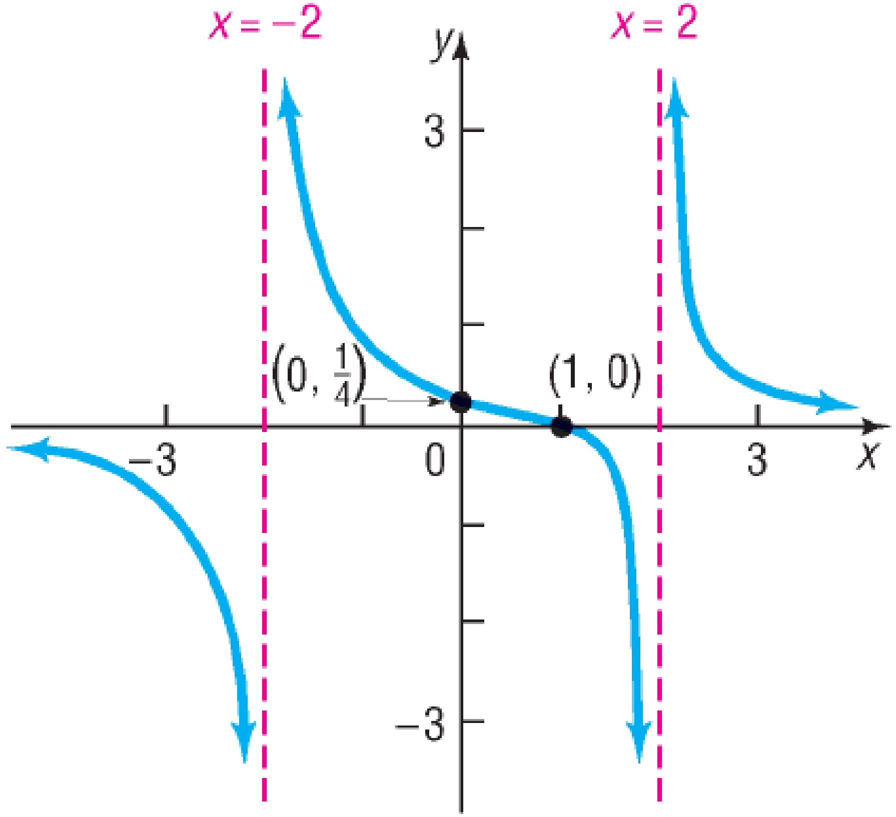

Step 8: Use the results obtained in Steps 1 through 7 to graph *R* by hand.

EXAMPLE

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph.

x-intercepts $p(x) = 0.$ y-intercept $R(0)$

STEP 4: Test for symmetry.

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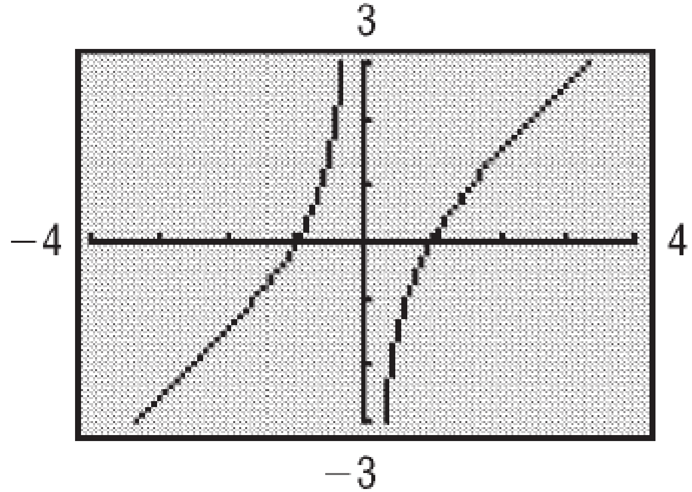

Step 5: Locate the vertical asymptotes.

Step 6: Locate the horizontal or oblique asymptotes, if any.

Step 7: Graph R using a graphing utility.

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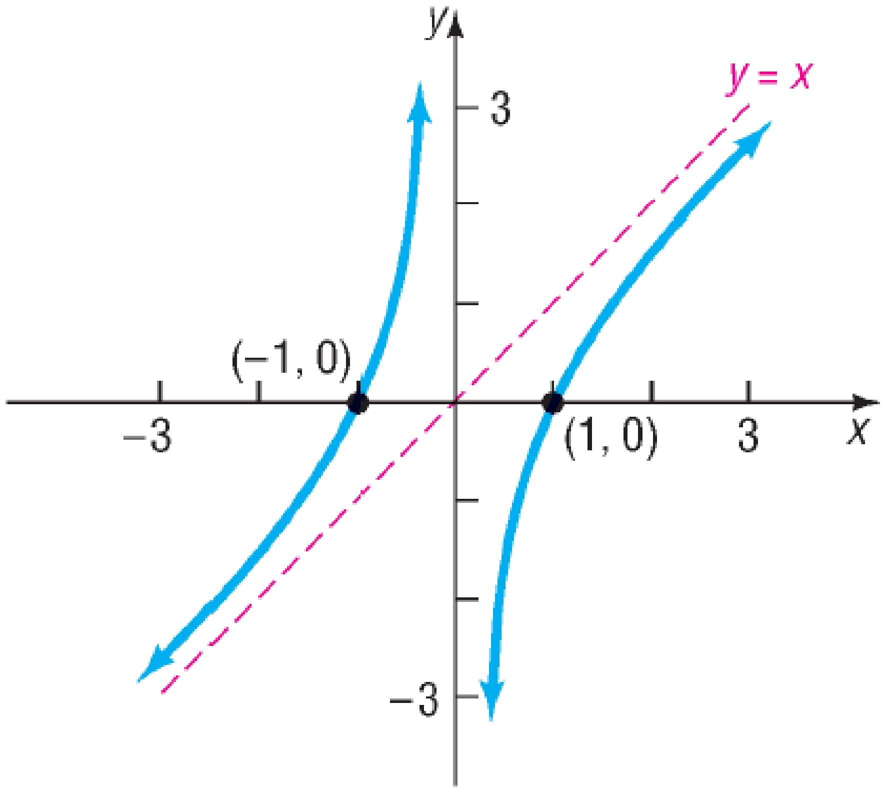

Step 8: Use the results obtained in Steps 1 through 7 to graph R by hand.

EXAMPLE

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

$$R(x) = \frac{3x(x-1)}{(x+4)(x-3)}$$

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph.

x-intercepts $p(x) = 0$ y-intercept $R(0)$

STEP 4: Test for symmetry.

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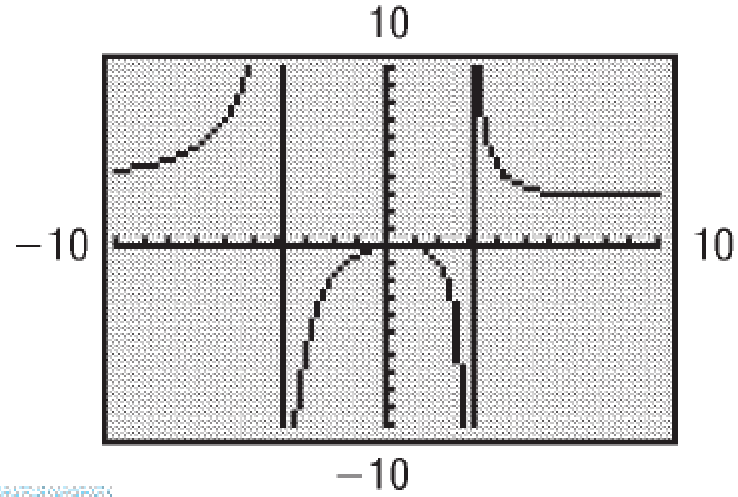

Connected mode

STEP 5: Locate the vertical asymptotes.

STEP 6: Locate the horizontal or oblique asymptotes, if any.

STEP 7: Graph R using a graphing utility.

EXAMPLE

Analyzing the Graph of a Rational Function

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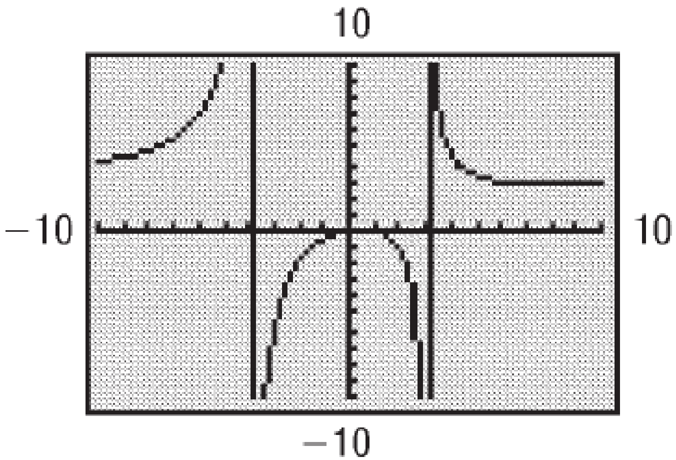
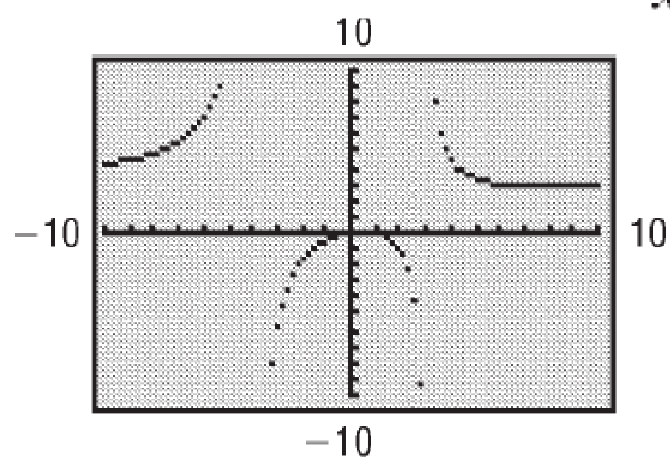
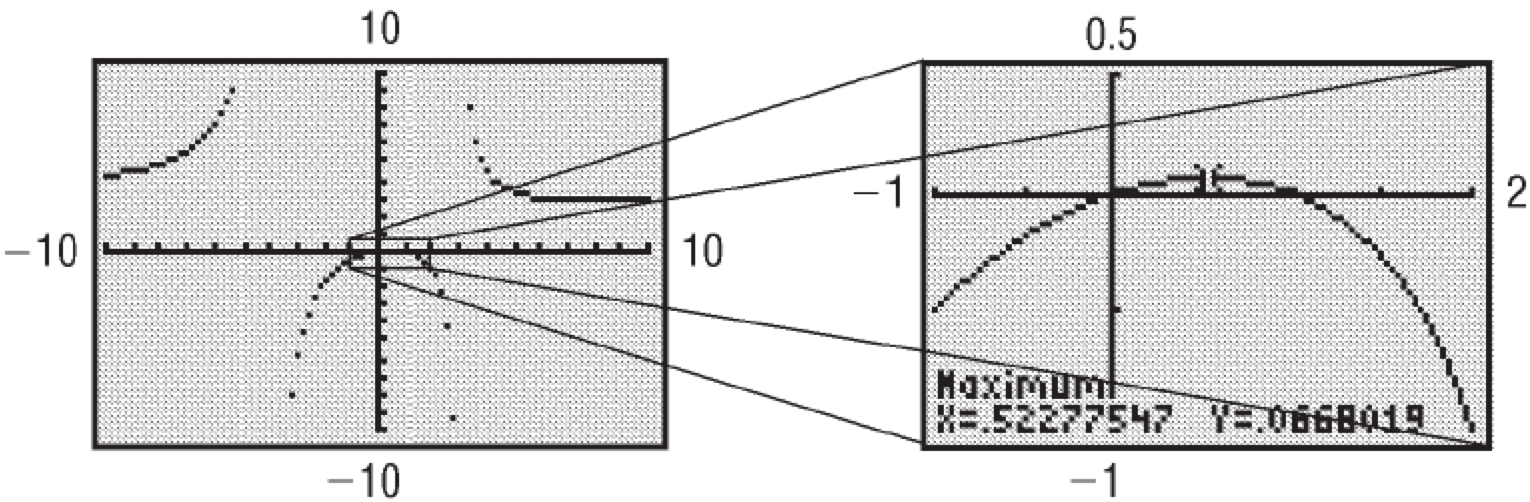


Connected mode



Dot mode



EXAMPLE

Analyzing the Graph of a Rational Function

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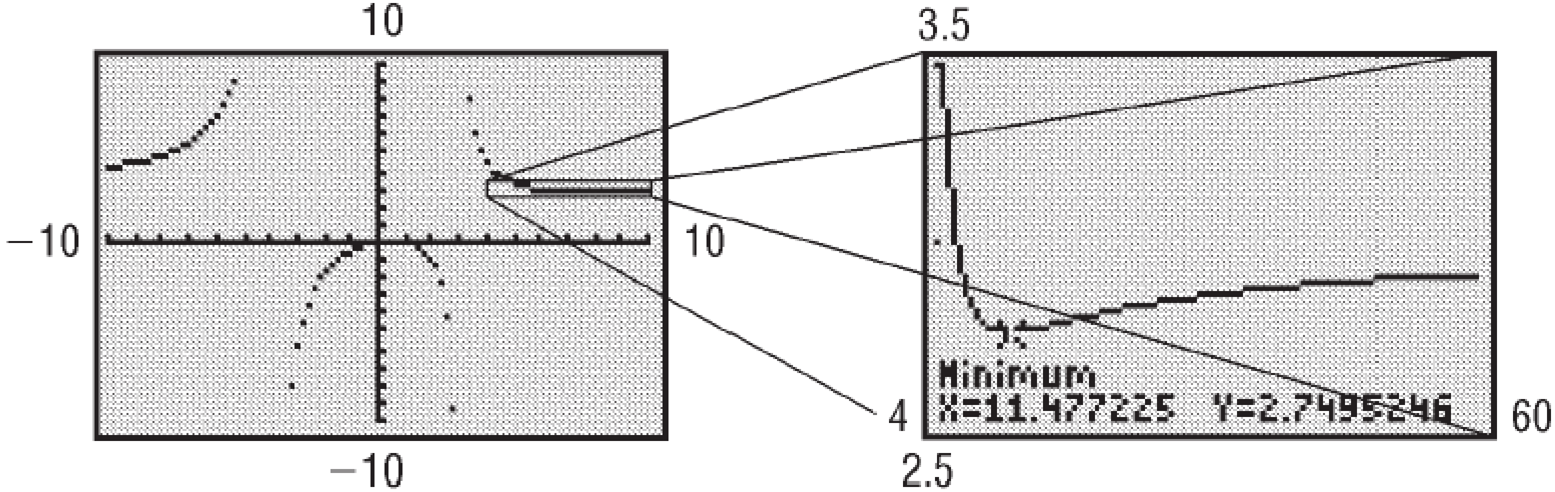
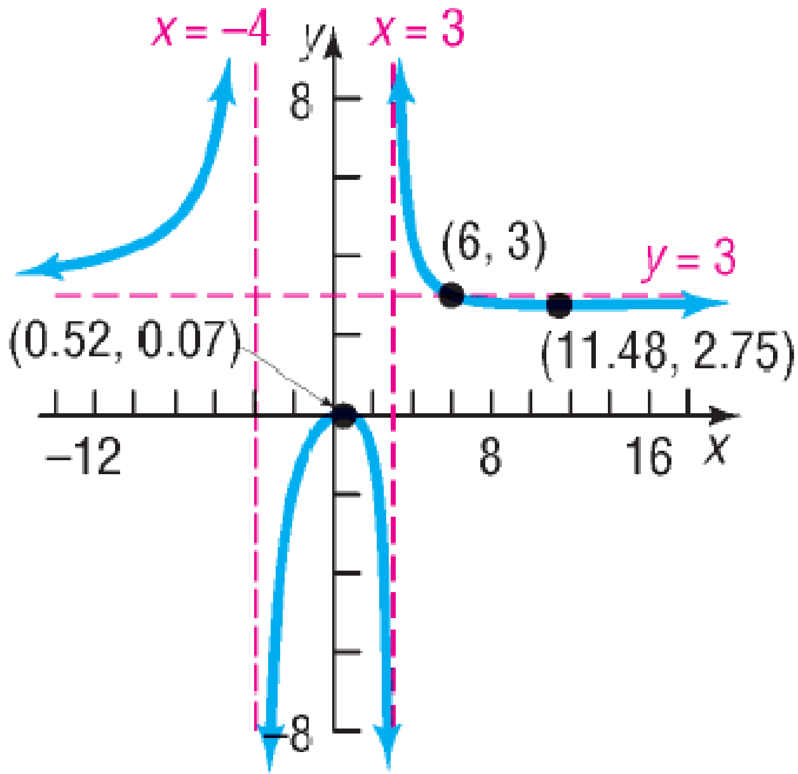


EXAMPLE

Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$



Step 8: Use the results obtained in Steps 1 through 7 to graph R by hand.

EXAMPLE

Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

$$R(x) = \frac{(2x-1)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

STEP 1: Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph.

x-intercepts $p(x) = 0$ y-intercept $R(0)$

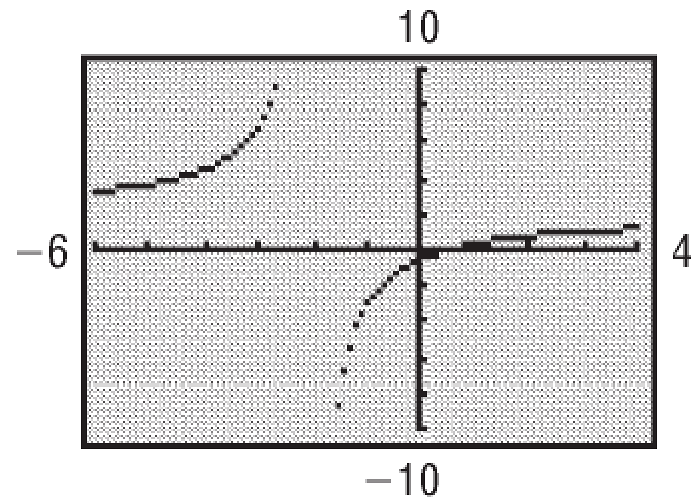
STEP 4: Test for symmetry.

EXAMPLE

Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

$$R(x) = \frac{2x - 1}{x + 2}$$



Step 5: Locate the vertical asymptotes.

Step 6: Locate the horizontal or oblique asymptotes, if any.

Step 7: Graph R using a graphing utility.

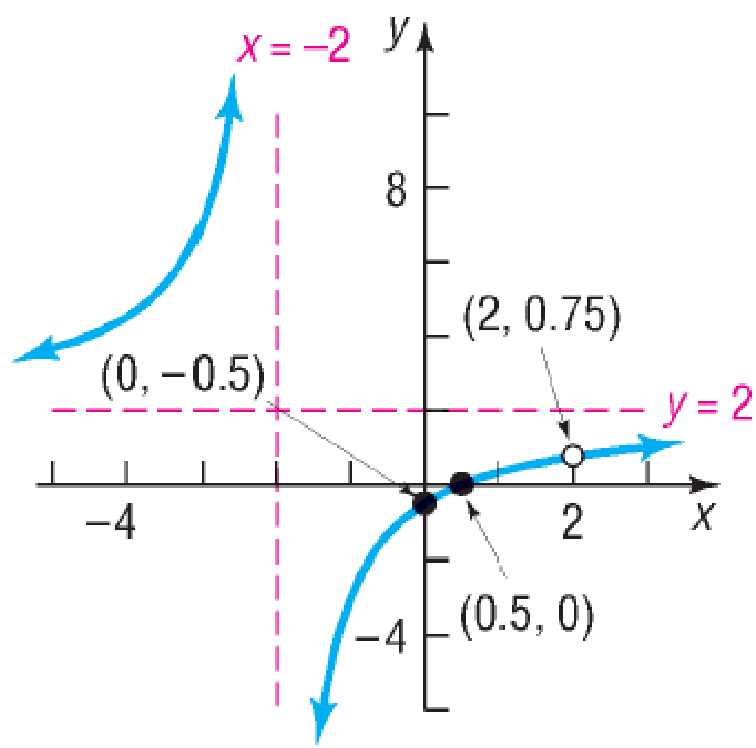
EXAMPLE

Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

X	Y1	
1.99	.74687	
1.999	.74969	
1.9999	.74997	
2	ERROR	
2.0001	.75003	
2.001	.75031	
2.01	.75312	

Y1 = (2X²−5X+2)/(...

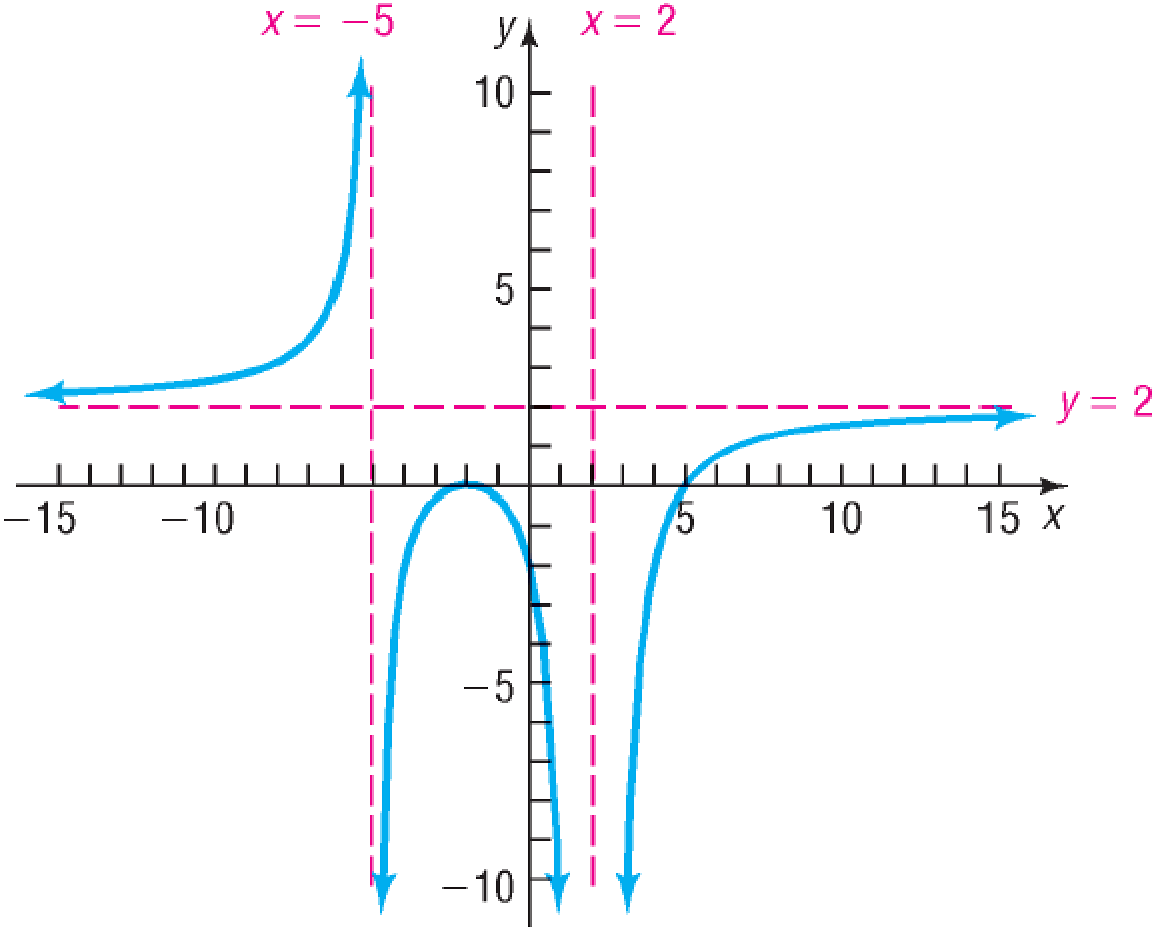


Step 8: Use the results obtained in Steps 1 through 7 to graph R by hand.

EXAMPLE

Constructing a Rational Function from Its Graph

Make up a rational function that might have the graph shown in Figure



OBJECTIVE 2

2 Solve Applied Problems Involving Rational Functions

EXAMPLE Finding the Least Cost of a Can

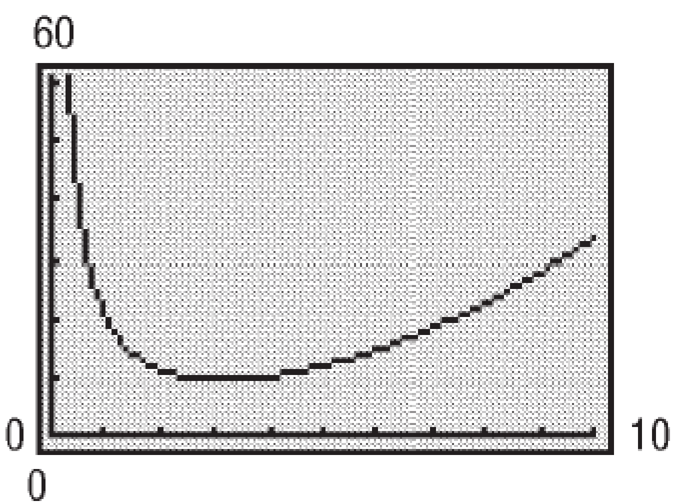
Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2} \text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

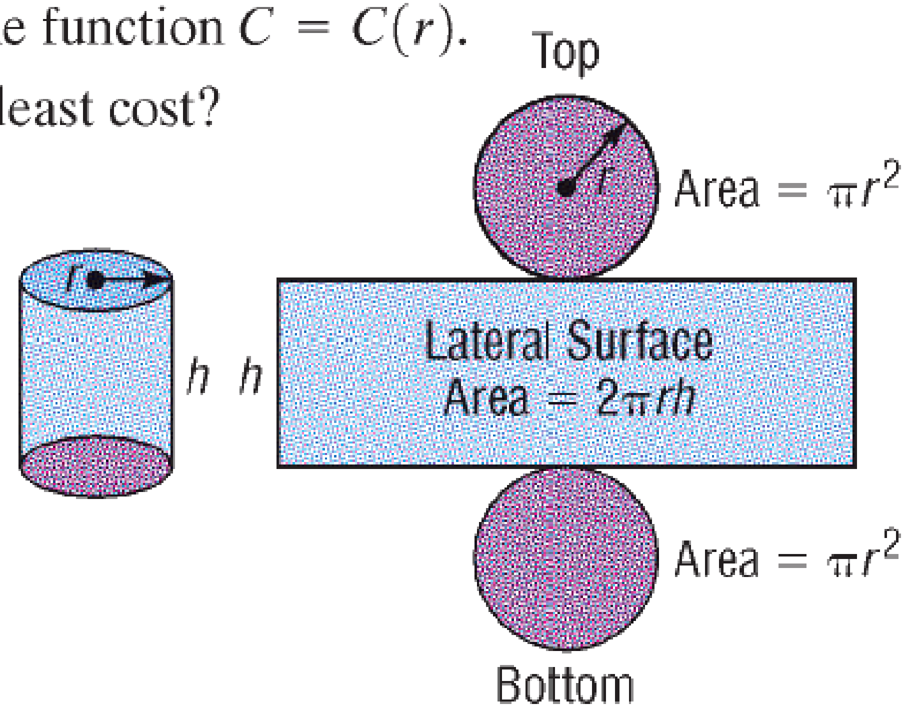
(a) Express the cost of material for the can as a function of the radius r of the can.

(b) Use a graphing utility to graph the function $C = C(r)$.

(c) What value of r will result in the least cost?

(d) What is this least cost?





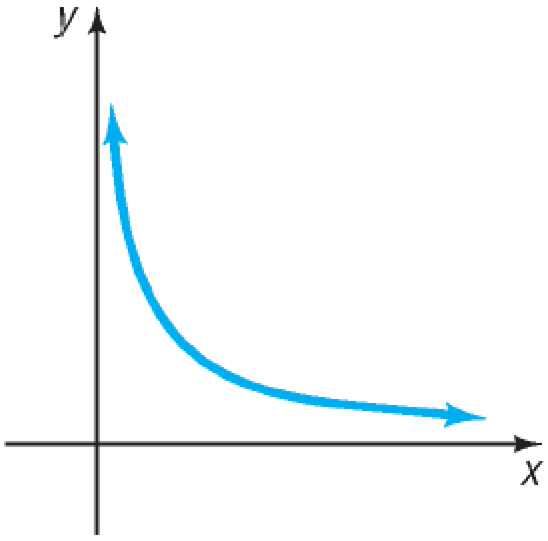
OBJECTIVE 3

3 Construct a Model Using Inverse Variation

Let x and y denote two quantities. Then y **varies inversely** with x, or y is **inversely proportional to** x, if there is a nonzero constant k such that

$$y = \frac{k}{x}$$

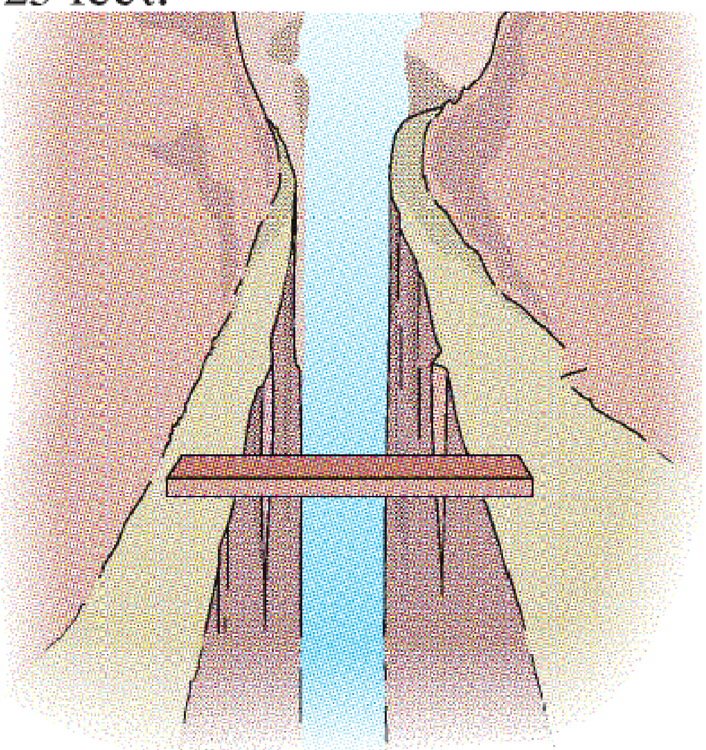
$y = \frac{k}{x}, k > 0, x > 0$

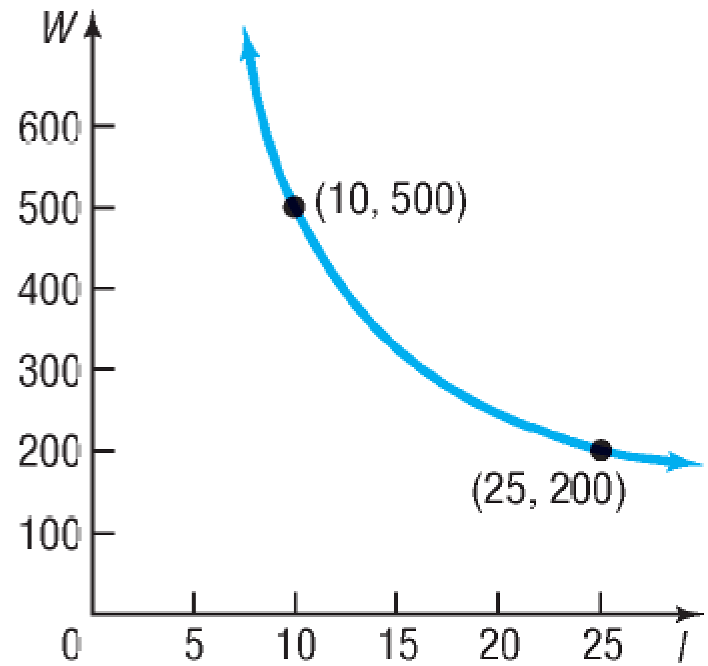


EXAMPLE

Maximum Weight That Can Be Supported by a Piece of Pine

See Figure 60. The maximum weight W that can be safely supported by a 2-inch by 4-inch piece of pine varies inversely with its length l. Experiments indicate that the maximum weight that a 10-foot-long pine 2-by-4 can support is 500 pounds. Write a general formula relating the maximum weight W (in pounds) to the length l (in feet). Find the maximum weight W that can be safely supported by a length of 25 feet.





OBJECTIVE 4

4 **Construct a Model Using Joint or Combined Variation**

EXAMPLE Loss of Heat Through a Wall

The loss of heat through a wall varies jointly with the area of the wall and the difference between the inside and outside temperatures and varies inversely with the thickness of the wall. Write an equation that relates these quantities.

$$L = k\frac{AT}{d}$$

EXAMPLE Force of the Wind on a Window

See Figure 62. The force F of the wind on a flat surface positioned at a right angle to the direction of the wind varies jointly with the area A of the surface and the square of the speed v of the wind. A wind of 30 miles per hour blowing on a window measuring 4 feet by 5 feet has a force of 150 pounds. What is the force on a window measuring 3 feet by 4 feet caused by a wind of 50 miles per hour?

$$F = kAv^2$$