Section 3.6

The Real Zeros of a Polynomial Function

OBJECTIVE 1

1 Use the Remainder and Factor Theorems

Theorem

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\text{dividend}}{f(x)} = \underset{\text{quotient}}{q(x)}\,\underset{\text{divisor}}{g(x)} + \underset{\text{remainder}}{r(x)} \qquad \textbf{(1)}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

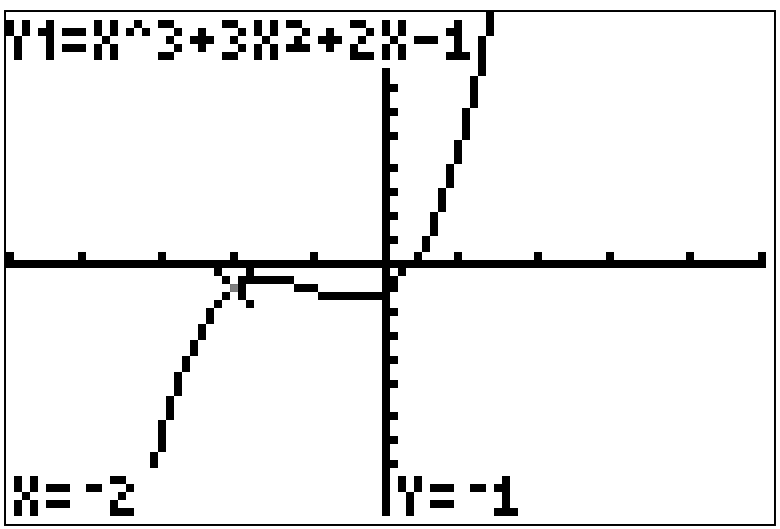
Remainder Theorem

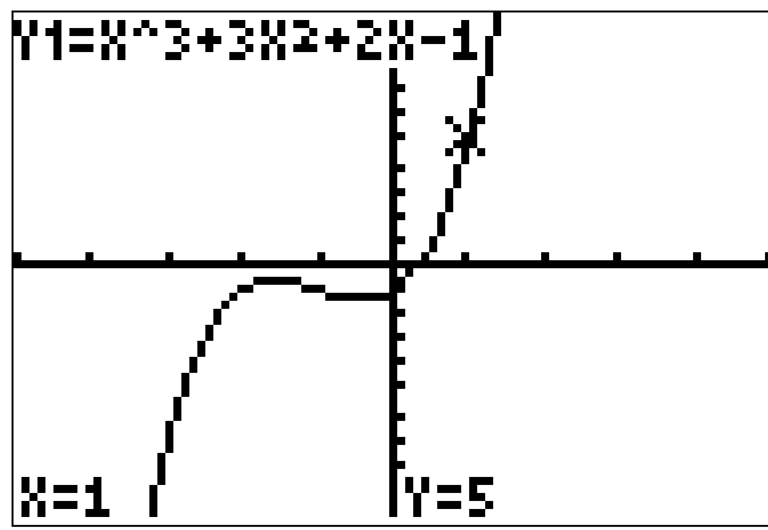
Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE Using the Remainder Theorem

Find the remainder if $f(x) = x^3 + 3x^2 + 2x - 1$ is divided by

(a) $x + 2$ (b) $x - 1$





Factor Theorem

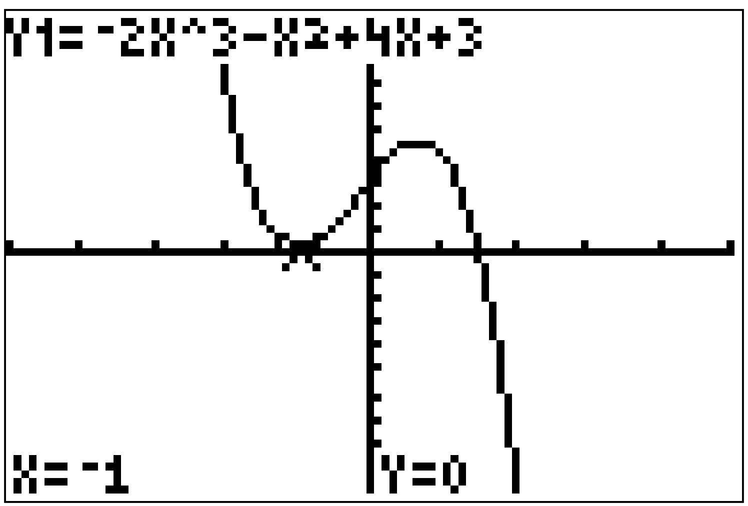
Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

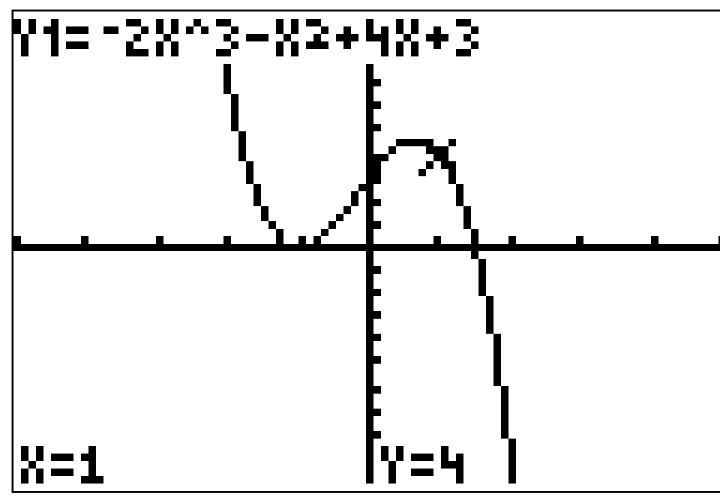
1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

EXAMPLE Using the Factor Theorem

Use the Factor Theorem to determine whether the function $f(x) = -2x^3 - x^2 + 4x + 3$ has the factor

(a) $x+1$ (b) $x-1$





Theorem

Number of Real Zeros

A polynomial function of degree n, $n \geq 1$, has at most n real zeros.

OBJECTIVE 2

2 Use the Rational Zeros Theorem

Theorem

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

EXAMPLE **Listing Potential Rational Zeros**

List the potential rational zeros of

$$f(x) = 3x^3 + 8x^2 - 7x - 12$$

$$\frac{\text{Factors of the constant}}{\text{Factors of the leading coefficient}}$$

$$p: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

$$q: \pm 1, \ \pm 3$$

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \ \pm \frac{4}{3}$$

OBJECTIVE 3

3 Find the Real Zeros of a Polynomial Function

Steps for Finding the Real Zeros of a Polynomial Function

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Step 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

Step 3: Using a graphing utility, graph the polynomial function.

Step 4: (a) Use eVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.

(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE

Finding the Rational Zeros of a Polynomial Function

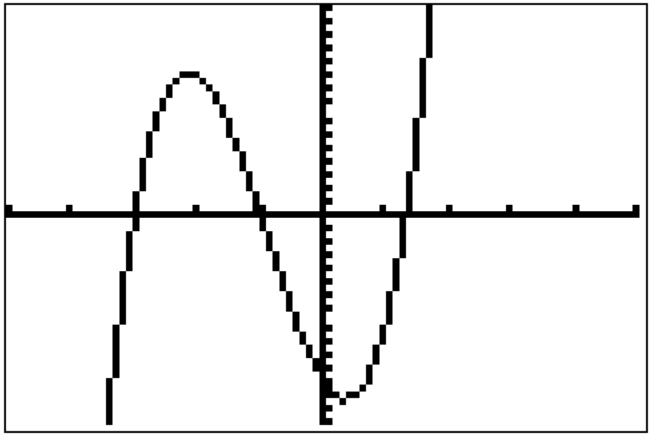
Find the rational zeros of the polynomial in the last example.

$f(x) = 3x^3 + 8x^2 - 7x - 12$ $\quad \frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Using a graphing utility, graph the polynomial function.



EXAMPLE

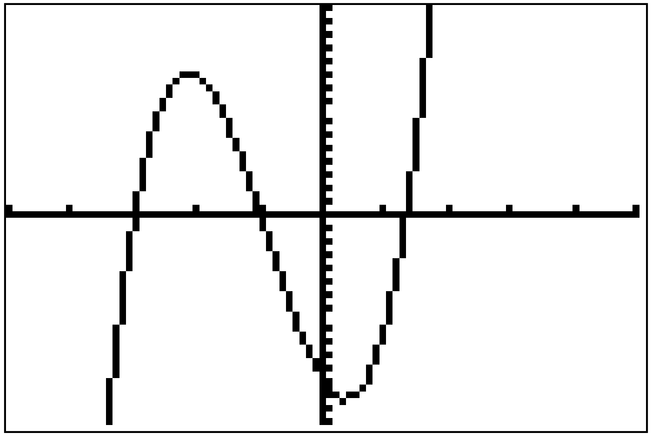
Finding the Rational Zeros of a Polynomial Function

Find the rational zeros of the polynomial in the last example.

$f(x) = 3x^3 + 8x^2 - 7x - 12$ $\quad \frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

STEP 4: (a) Use eVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.

(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).



EXAMPLE

Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = 2x^4 + 13x^3 + 29x^2 + 27x + 9$.

Write f in factored form.

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Using a graphing utility, graph the polynomial function.

$$\frac{p}{q}: \pm 1,\ \pm\frac{1}{2},\ \pm 3,\ \pm\frac{3}{2},\ \pm 9,\ \pm\frac{9}{2}$$

EXAMPLE

Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = 2x^4 + 13x^3 + 29x^2 + 27x + 9$.

Write f in factored form. $f(x) = (x+1)^2(2x+3)(x+3)$

STEP 4: (a) Use eVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.

(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

$$\frac{p}{q}: \pm 1,\ \pm\frac{1}{2},\ \pm 3,\ \pm\frac{3}{2},\ \pm 9,\ \pm\frac{9}{2}$$

OBJECTIVE 4

4 Solve Polynomial Equations

EXAMPLE Solving a Polynomial Equation

Solve the equation: $2x^4 + 13x^3 + 29x^2 + 27x + 9 = 0$

$$(x+1)^2(2x+3)(x+3) = 0$$

Theorem

Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

Corollary

A polynomial function (with real coefficients) of odd degree has at least one real zero.

OBJECTIVE 5

5 **Use the Theorem for Bounds on Zeros**

BOUND

$$-M \le \text{any zero of } f \le M$$

Theorem

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

A bound M on the zeros of f is the smaller of the two numbers

$$\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, \quad 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} \qquad \textbf{(4)}$$

where Max { } means "choose the largest entry in { }."

EXAMPLE

Using the Theorem for Finding Bounds on Zeros

Find a bound to the zeros of each polynomial.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

$$\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}$$

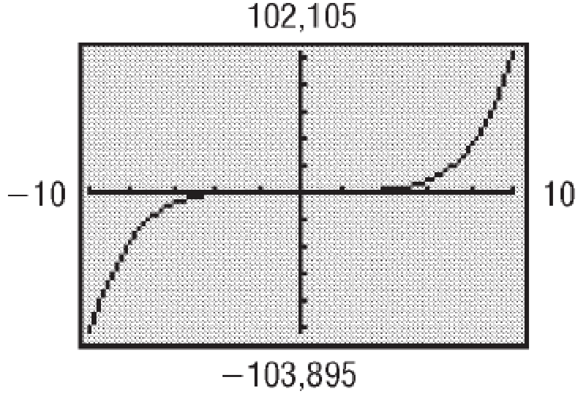
$$1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$$

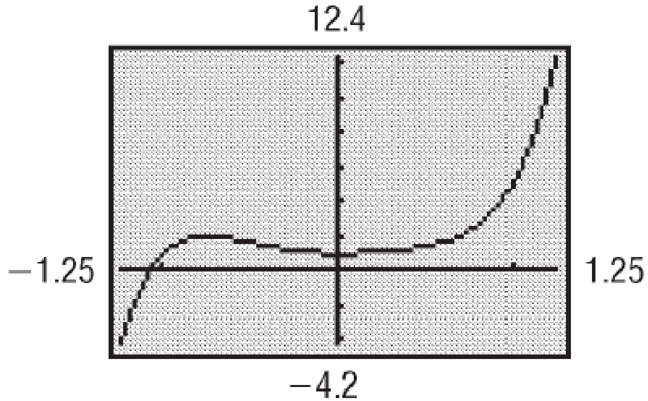
EXAMPLE

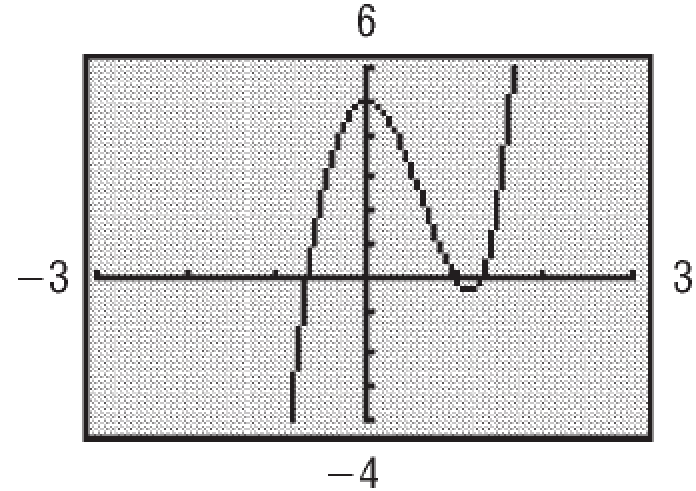
Obtaining Graphs Using Bounds on Zeros

Obtain a graph for each polynomial.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$







EXAMPLE

Finding the Zeros of a Polynomial

Find all the real zeros of the polynomial function

$$f(x) = x^5 - 1.8x^4 - 17.79x^3 + 31.672x^2 + 37.95x - 8.7121$$

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Using a graphing utility, graph the polynomial function. To aid in this, first use the bounds theorem to set Xmin andXmax

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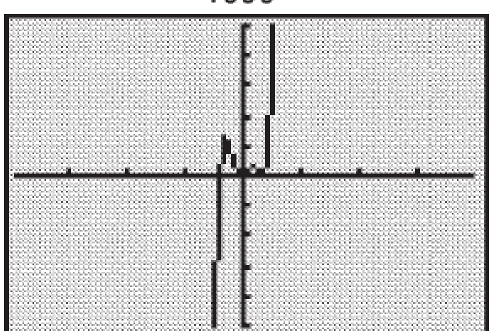
−38.95  38.95

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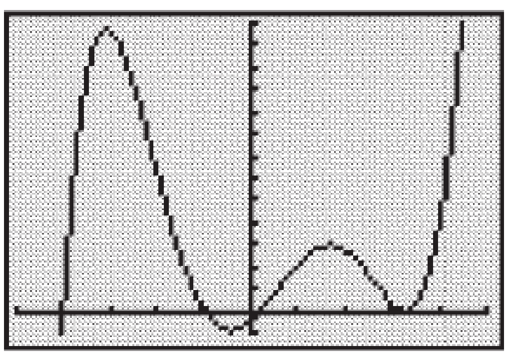
−5  5

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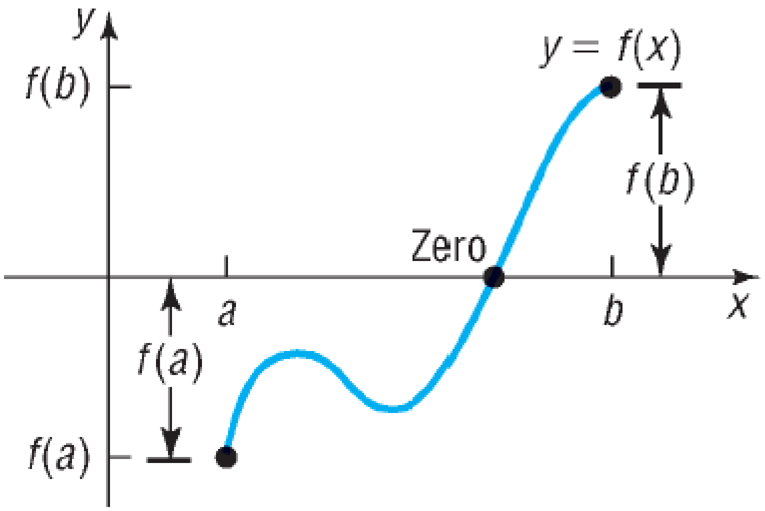
OBJECTIVE 6

6 Use the Intermediate Value Theorem

Intermediate Value Theorem

Let f denote a continuous function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then f has at least one zero between a and b.



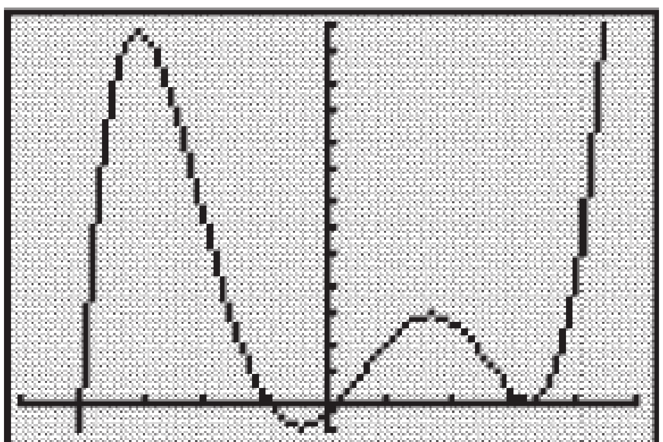
If $f(a) < 0$ and $f(b) > 0$ and if f is continuous, there is at least one zero between a and b.

EXAMPLE

Using the Intermediate Value Theorem and a Graphing Utility to Locate Zeros

Using the function from the last example, determine whether there is a repeated zero or two distinct zeros near 3.30.

$$f(x) = x^5 - 1.8x^4 - 17.79x^3 + 31.672x^2 + 37.95x - 8.7121$$



X	Y1
3.27	.08567
3.28	.03829
3.29	.00956
3.3	$^{-}1E^{-}4$
3.31	.0097
3.32	.03936
3.33	.08931

Y1=X^5−1.8X^4−1...