Section 3.7

Complex Zeros;

Fundamental Theorem of Algebra

A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **(complex) zero** of f if $f(r) = 0$.

Fundamental Theorem of Algebra

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Theorem

Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \tag{2}$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n (not necessarily distinct) zeros.

OBJECTIVE 1

1 Use the Conjugate Pairs Theorem

Conjugate Pairs Theorem

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

COROLLARY

A polynomial f of odd degree with real coefficients has at least one real zero.

EXAMPLE

Using the Conjugate Pairs Theorem

A polynomial of degree 5 whose coefficients are real numbers has the zeros -2, -3*i*, and 2+4*i*. Find the remaining two zeros.

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

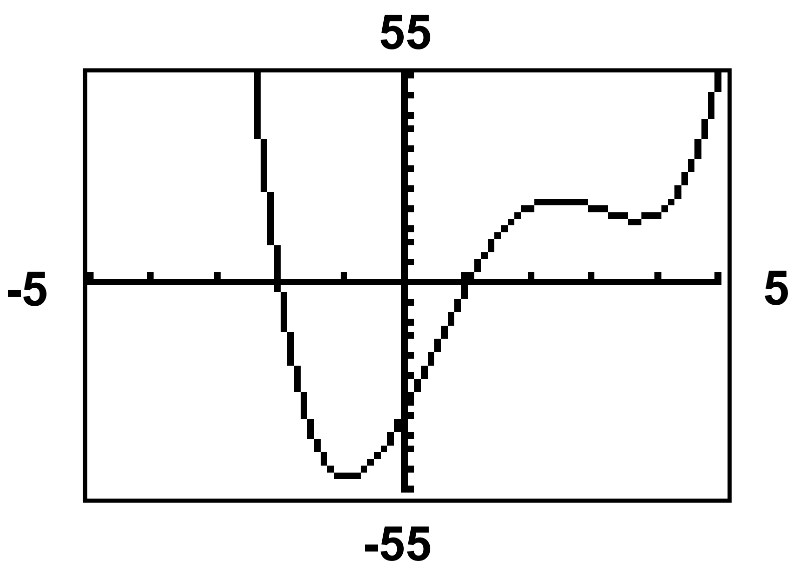
OBJECTIVE 2

2 Find a Polynomial Function with Specified Zeros

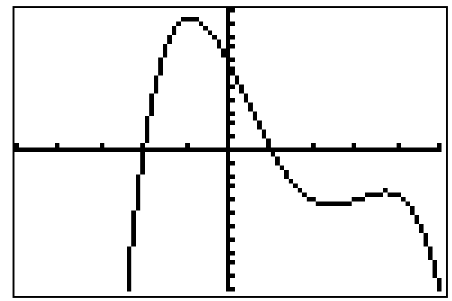
EXAMPLE

Finding a Polynomial Function Whose Zeros Are Given

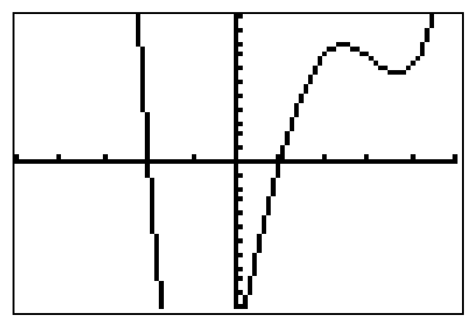
Find a polynomial *f* of degree 4 whose coefficients are real numbers and that has the zeros -2, 1, 4+*i*. Graph the polynomial.



For leading coefficient of 1



For leading coefficient of -1



For leading coefficient of 2

Theorem

Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.

OBJECTIVE 3

3 Find the Complex Zeros of a Polynomial

EXAMPLE

Finding the Complex Zeros of a Polynomial

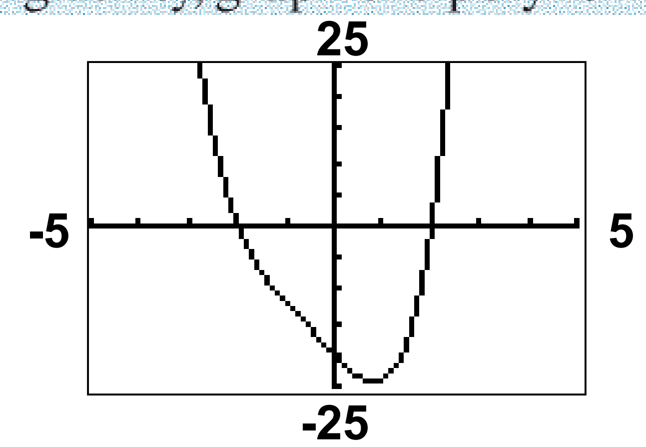
Find the complex zeros of the polynomial function

$x^4 + 2x^3 + x^2 - 8x - 20$ $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

STEP 1: Use the degree of the polynomial to determine the maximum number of zeros.

STEP 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Using a graphing utility, graph the polynomial function.



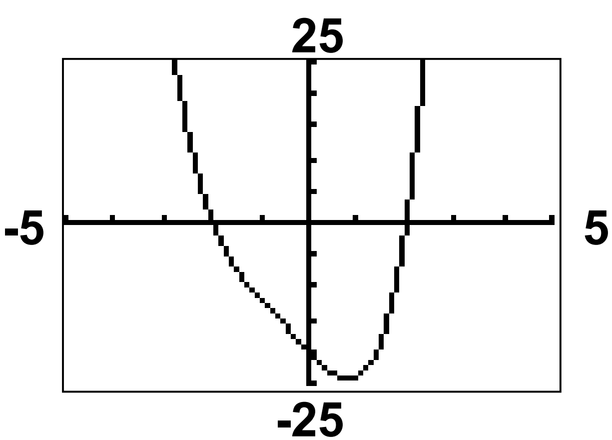
EXAMPLE

Finding the Complex Zeros of a Polynomial

Find the complex zeros of the polynomial function

$x^4 + 2x^3 + x^2 - 8x - 20$ $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Step 4: (a) Use eVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.
(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).



$$(x+2)(x-2)(x^2+2x+5)$$