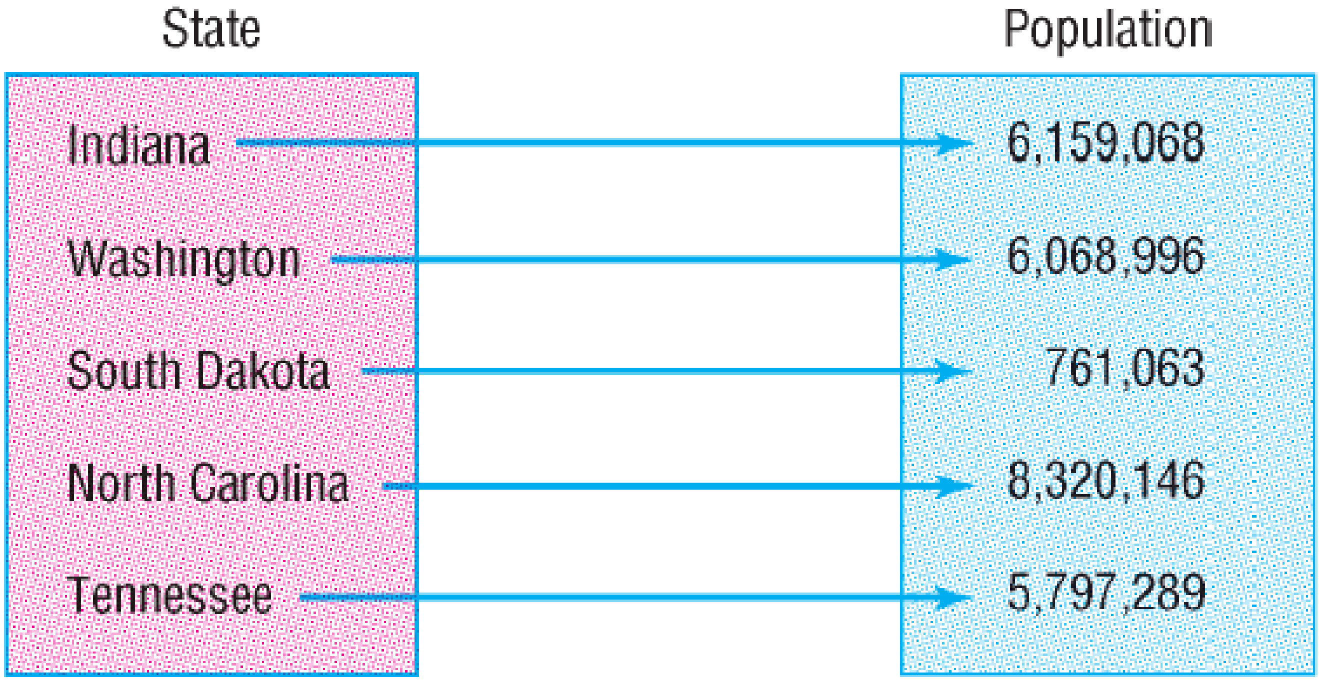
Section 4.2

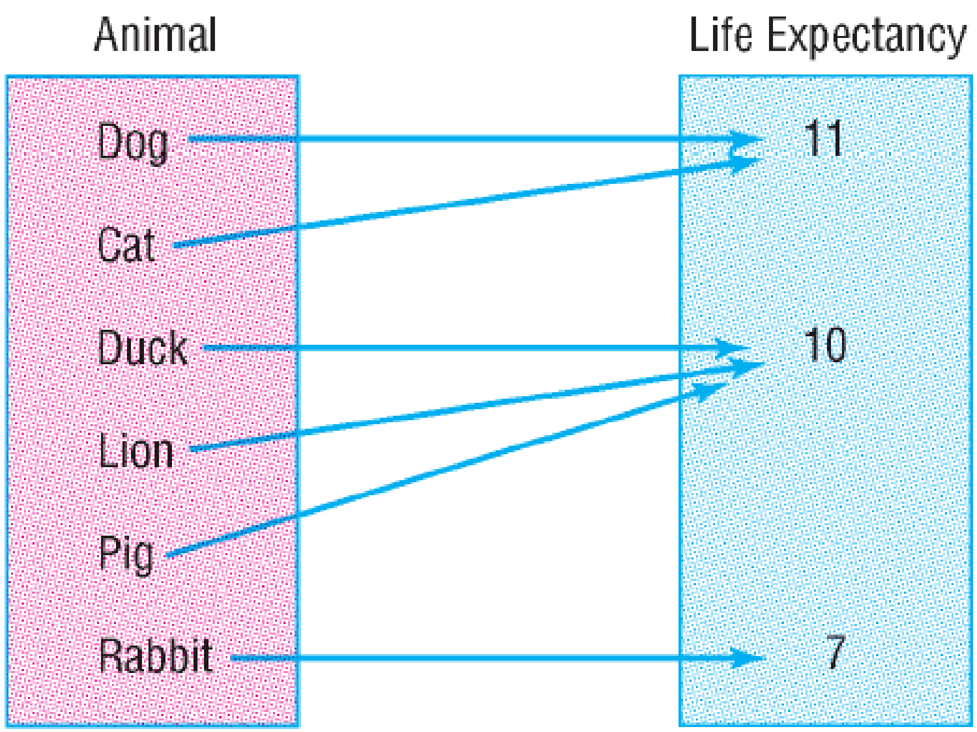
One-to-One Functions;

Inverse Functions

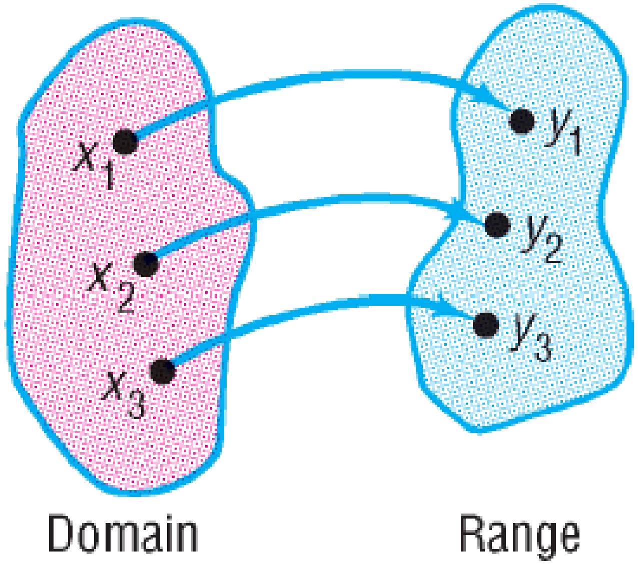
OBJECTIVE 1

1 Determine Whether a Function Is One-to-One

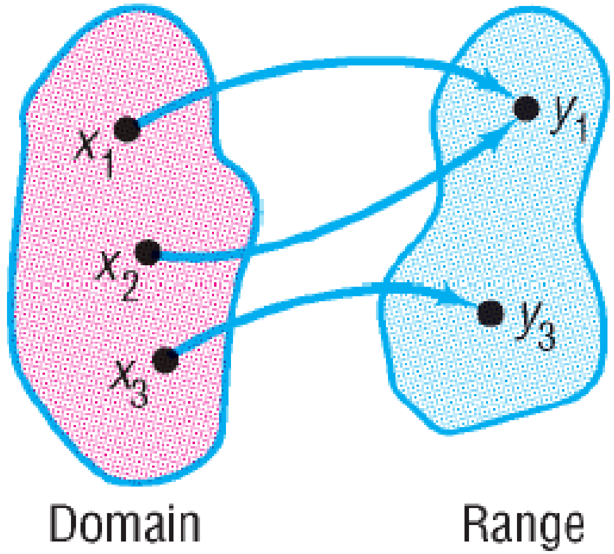

State
Population
Indiana
Washington
South Dakota
North Carolina
Tennessee
6,159,068
6,068,996
761,063
8,320,146
5,797,289


Animal
Life Expectancy
Dog
Cat
Duck
Lion
Pig
Rabbit
11
10
7

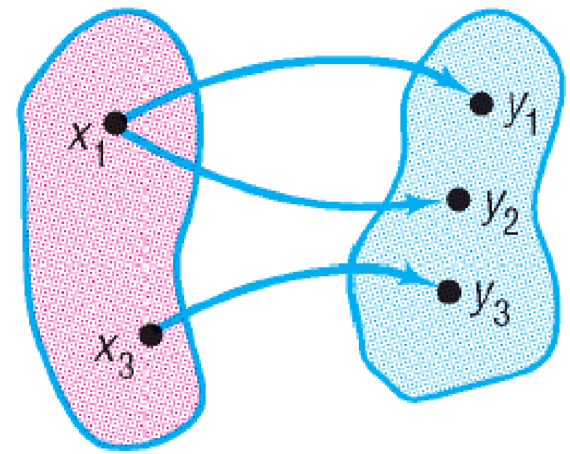
A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then $f(x_1) \neq f(x_2)$.



One-to-one function:
Each x in the domain has one and only one image in the range. No y in the range is the image of more than one x.



Not a one-to-one function:
y_1 is the image of both x_1 and x_2



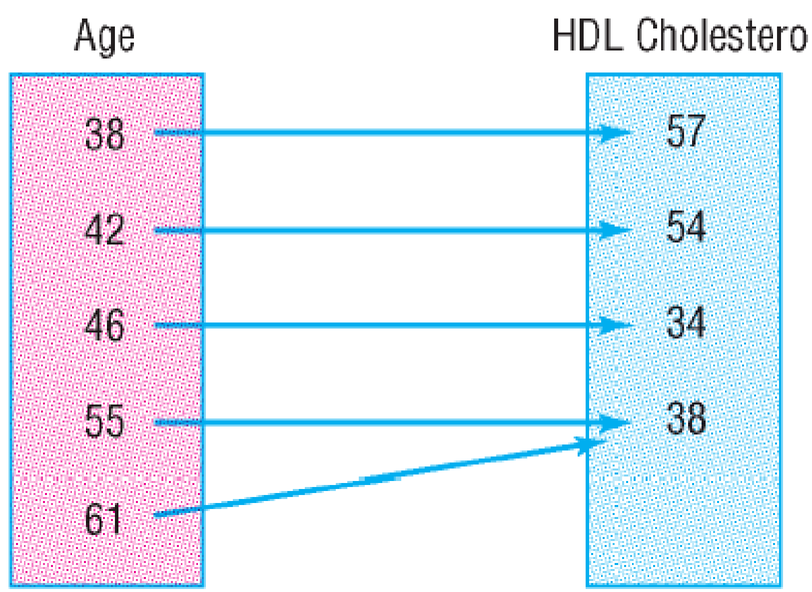
Not a function:
x_1 has two images, y_1 and y_2

EXAMPLE

Determining Whether a Function Is One-to-One

Determine whether the following functions are one-to-one.

(a) For the following function, the domain represents the age of five males and the range represents their HDL (good) cholesterol (mg/dL).

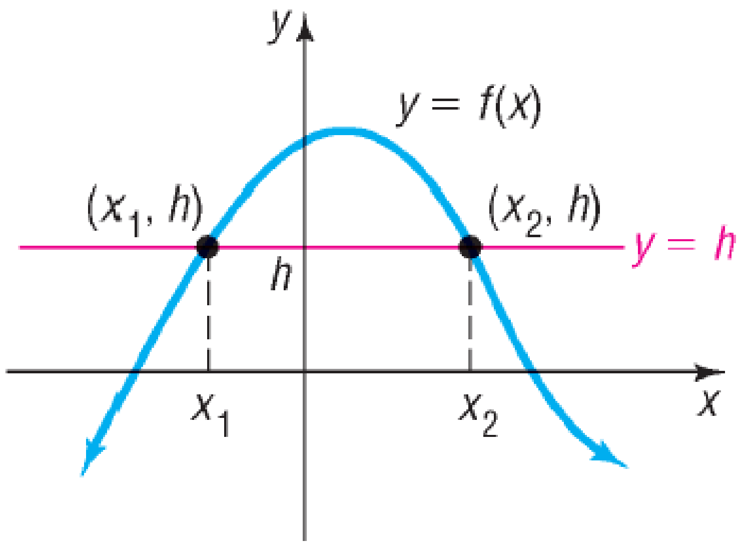


(b) $\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$

Theorem

Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

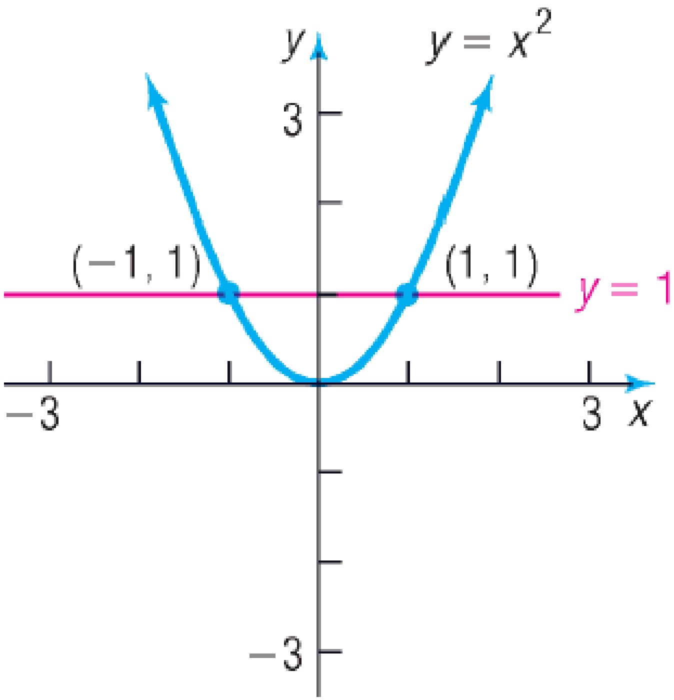


$f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$; f is not a one-to-one function.

EXAMPLE **Using the Horizontal-line Test**

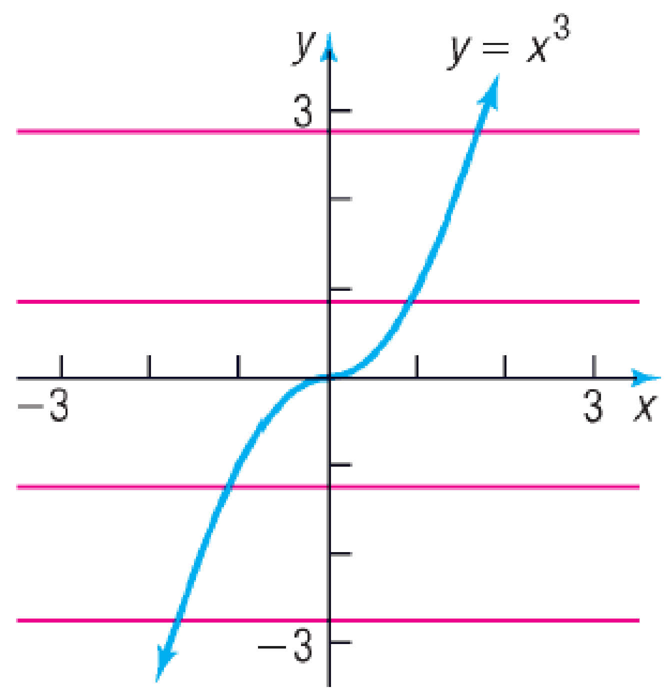
For each function, use the graph to determine whether the function is one-to-one.

(a) $f(x) = x^2$



A horizontal line intersects the graph twice; *f* is not one-to-one

(b) $g(x) = x^3$



Every horizontal line intersects the graph exactly once; *g* is one-to-one

Theorem

A function that is increasing on an interval *I* is a one-to-one function in *I*.

A function that is decreasing on an interval *I* is a one-to-one function on *I*.

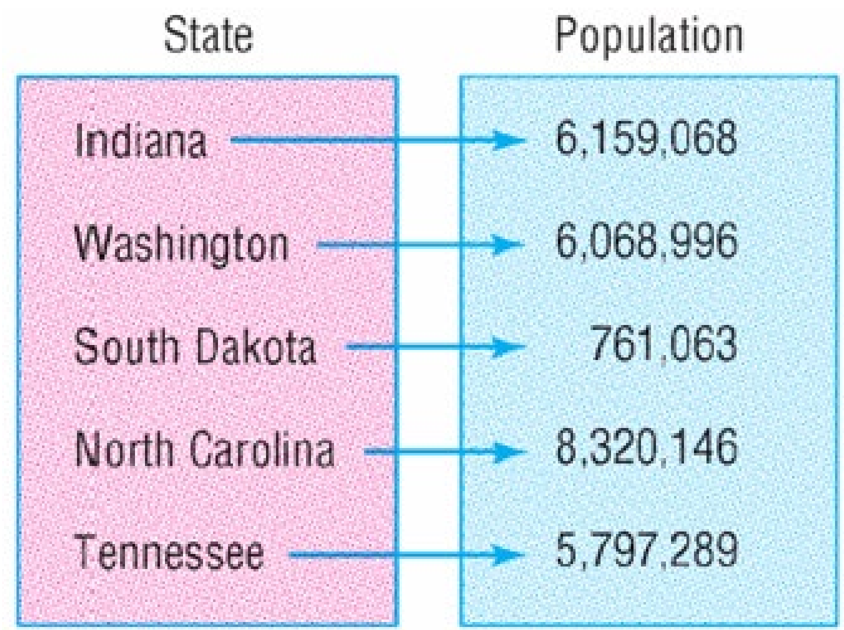
OBJECTIVE 2

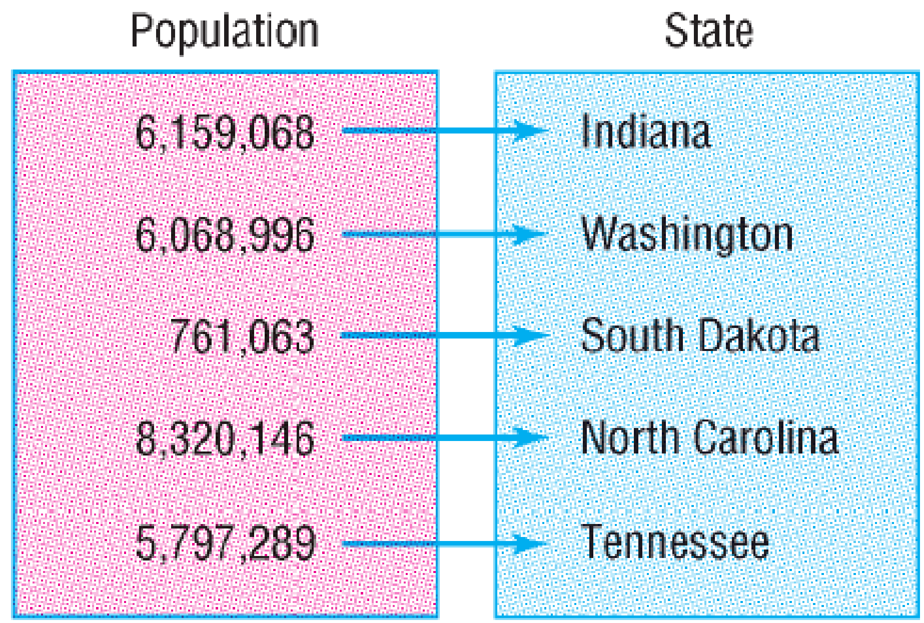
2 Determine the Inverse of a Function Defined by a Map or an Ordered Pair

EXAMPLE

Finding the Inverse of a Function Defined by a Map

Find the inverse of the following function. Let the domain of the function represent certain states, and let the range represent the state's population. State the domain and the range of the inverse function.



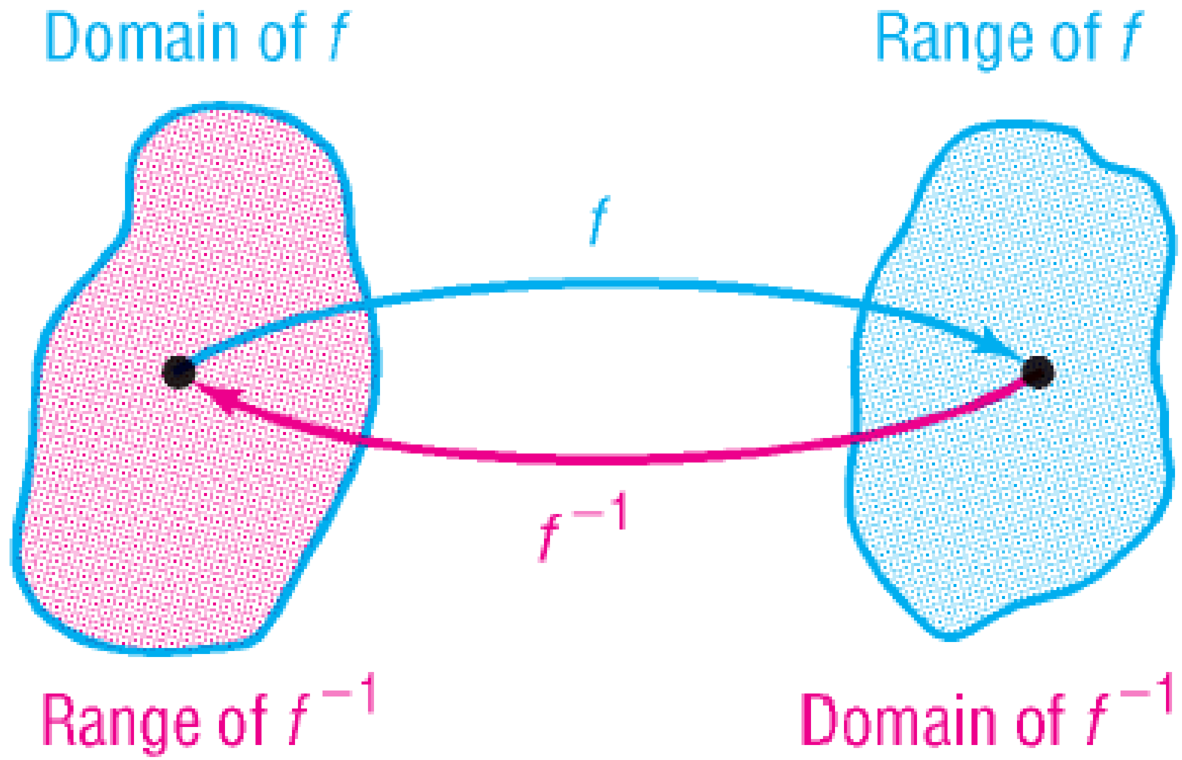


EXAMPLE

Finding the Inverse of a Function Defined By a Set of Ordered Pairs

Find the inverse of the following one-to-one function:

$$\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$



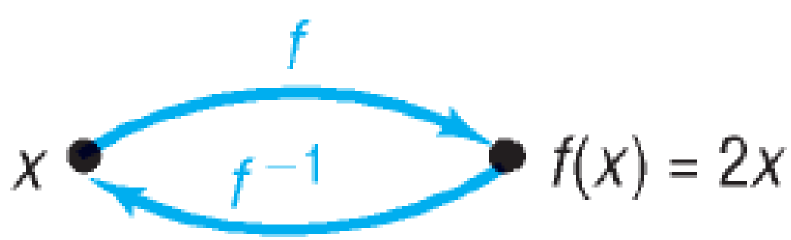
Domain of f = Range of f^{-1} Range of f = Domain of f^{-1}

$$\text{Input } x \xrightarrow{\text{Apply } f} f(x) \xrightarrow{\text{Apply } f^{-1}} f^{-1}(f(x)) = x$$

$$\text{Input } x \xrightarrow{\text{Apply } f^{-1}} f^{-1}(x) \xrightarrow{\text{Apply } f} f(f^{-1}(x)) = x$$

$f^{-1}(f(x)) = x$ where x is in the domain of f

$f(f^{-1}(x)) = x$ where x is in the domain of f^{-1}



$$f^{-1}(2x) = \frac{1}{2}(2x) = x$$

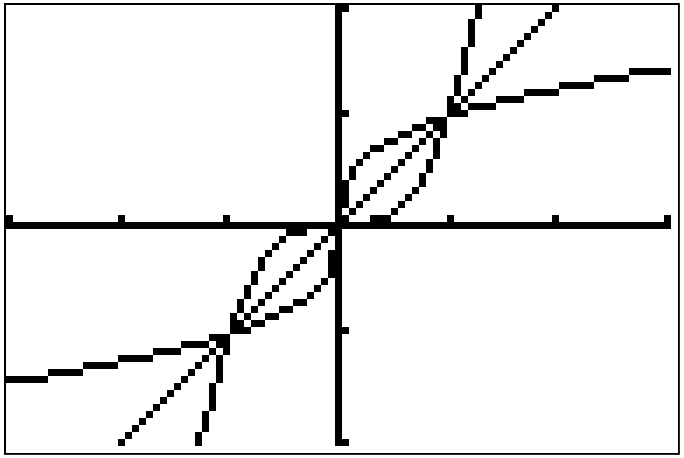
EXAMPLE **Verifying Inverse Functions**

(a) We verify that the inverse of $g(x) = x^3$ is $g^{-1}(x) = \sqrt[3]{x}$ by showing that

$$g^{-1}(g(x)) = g^{-1}(x^3) = \sqrt[3]{x^3} = x \quad \text{for all } x \text{ in the domain of } g$$

$$g(g^{-1}(x)) = g(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \quad \text{for all } x \text{ in the domain of } g^{-1}.$$

Exploration



Simultaneously graph $Y_1 = x$, $Y_2 = x^3$, and $Y_3 = \sqrt[3]{x}$ on a square screen with $-3 \le x \le 3$. What do you observe about the graphs of $Y_2 = x^3$, its inverse $Y_3 = \sqrt[3]{x}$, and the line $Y_1 = x$?

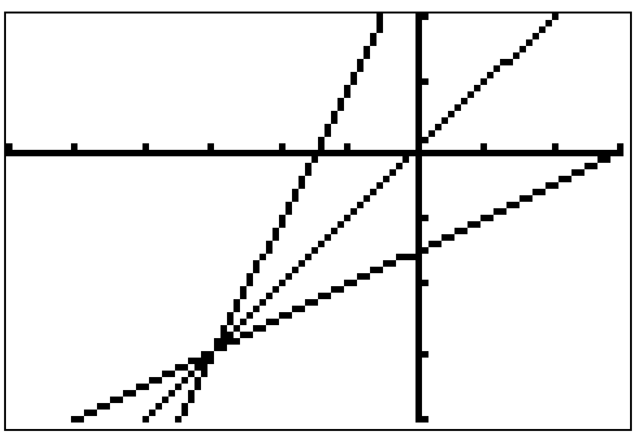
EXAMPLE Verifying Inverse Functions

(b) We verify that the inverse of $f(x) = 2x + 3$ is $f^{-1}(x) = \frac{1}{2}(x - 3)$ by showing that

$$f^{-1}(f(x)) = f^{-1}(2x + 3) = \frac{1}{2}[(2x + 3) - 3] = \frac{1}{2}(2x) = x \quad \text{for all } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = f\left(\frac{1}{2}(x - 3)\right) = 2\left[\frac{1}{2}(x - 3)\right] + 3 = (x - 3) + 3 = x \quad \text{for all } x \text{ in the domain of } f^{-1}.$$

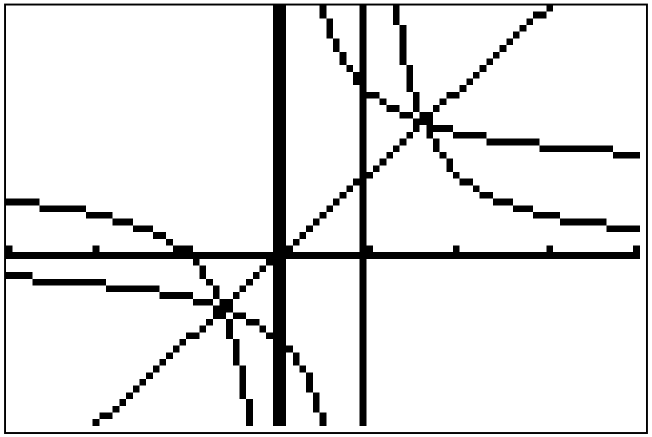
Exploration



Repeat this experiment by simultaneously graphing $Y_1 = x$, $Y_2 = 2x + 3$, and $Y_3 = \frac{1}{2}(x - 3)$ on a square screen with $-6 \le x \le 3$. Do you see the symmetry of the graph of Y_2 and its inverse Y_3 with respect to the line $Y_1 = x$?

EXAMPLE Verifying Inverse Functions

Verify that the inverse of $f(x) = \frac{1}{x-1}$ is $f^{-1}(x) = \frac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of x is $f(f^{-1}(x)) = x$?



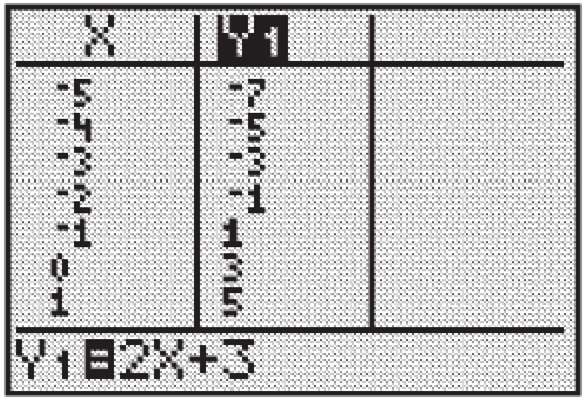
$f^{-1}(x) = \frac{1}{x} + 1$

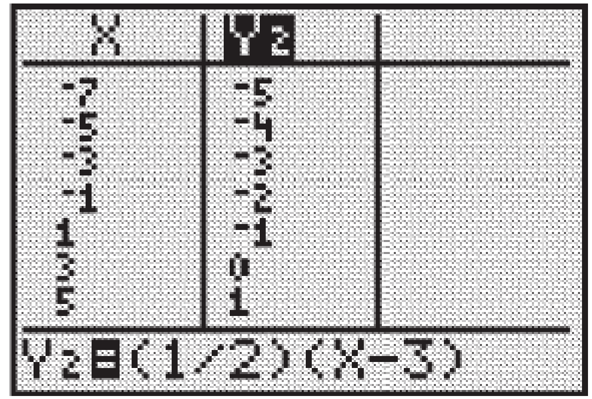
$f(x) = \frac{1}{x-1}$

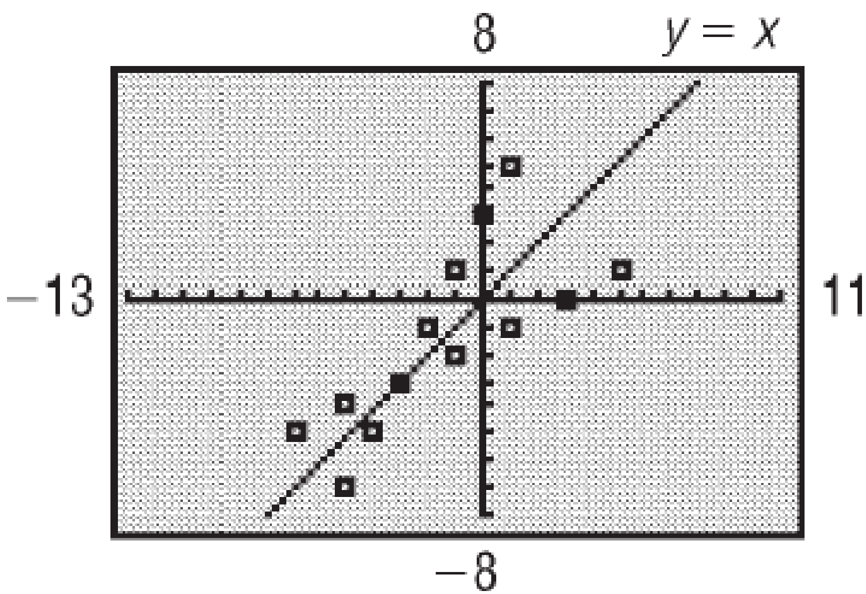
OBJECTIVE 3

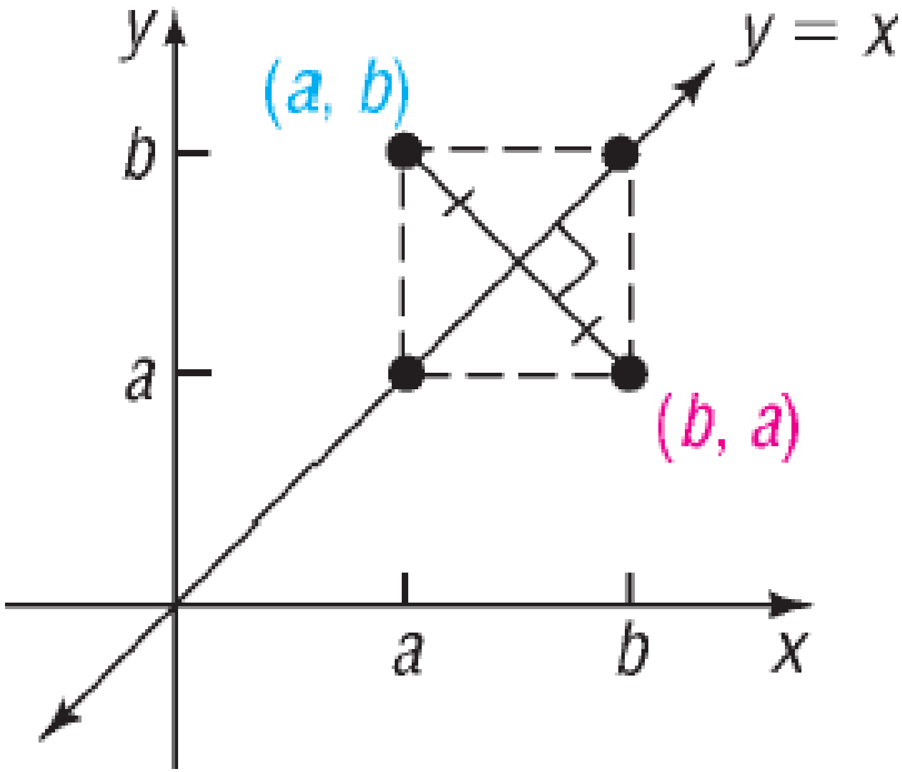
3 **Obtain the Graph of the Inverse Function from the Graph of the Function**

$$f(x) = 2x + 3 \quad f^{-1}(x) = \frac{1}{2}(x - 3)$$



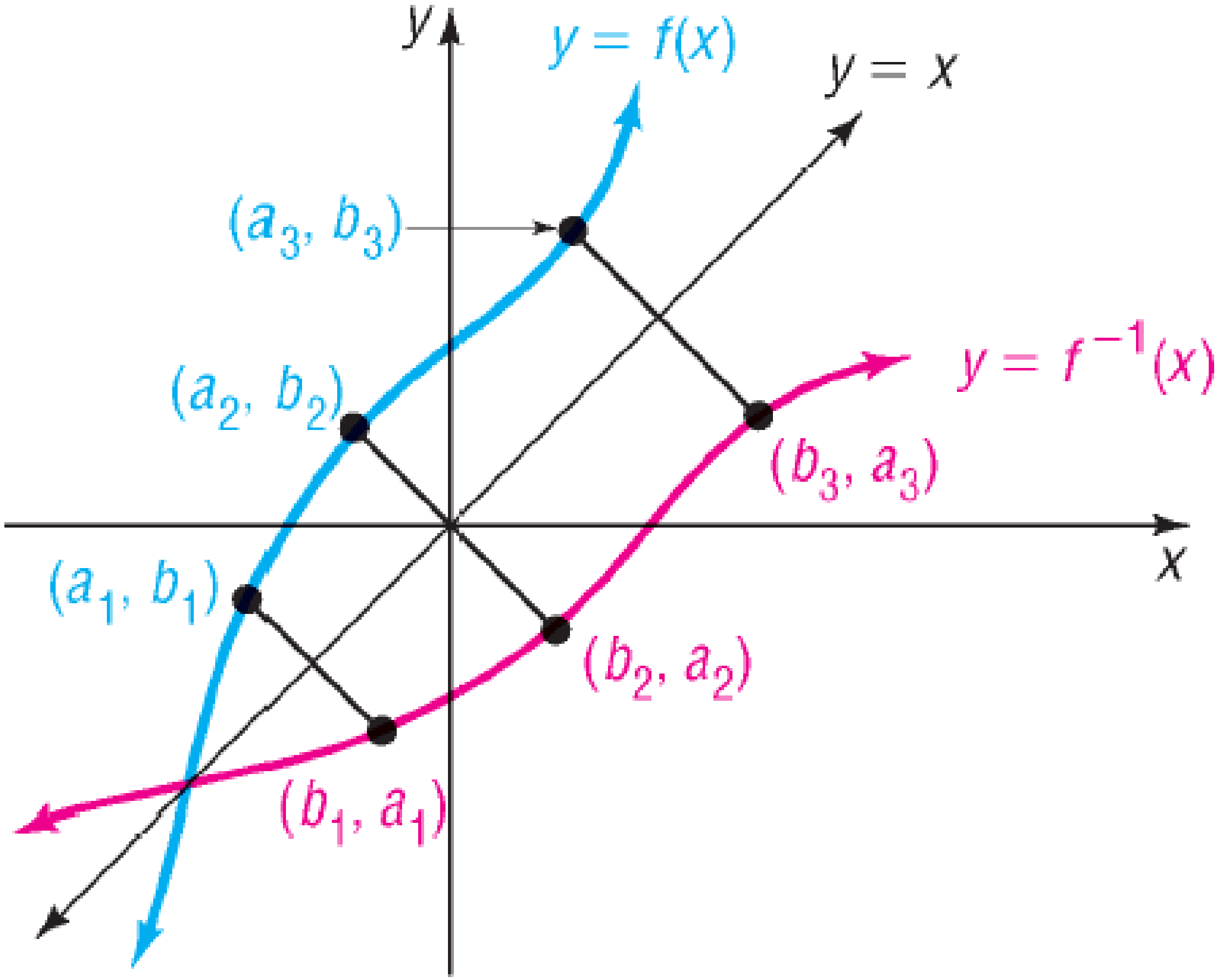






Theorem

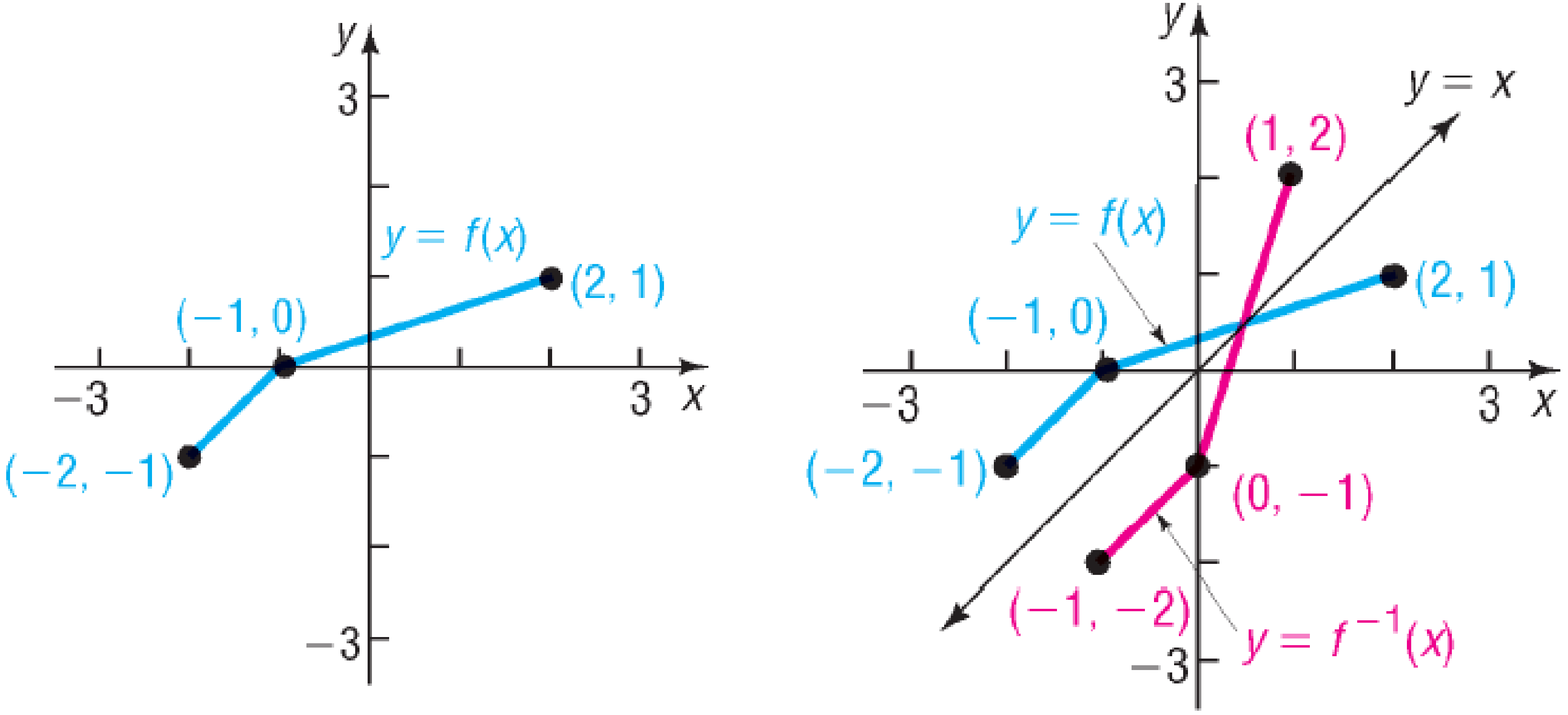
The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.



y
y = f(x)
y = x
(a3, b3)
(a2, b2)
y = f −1(x)
(b3, a3)
x
(a1, b1)
(b2, a2)
(b1, a1)

EXAMPLE Graphing the Inverse Function

The graph in Figure 16(a) is that of a one-to-one function $y = f(x)$. Draw the graph of its inverse.

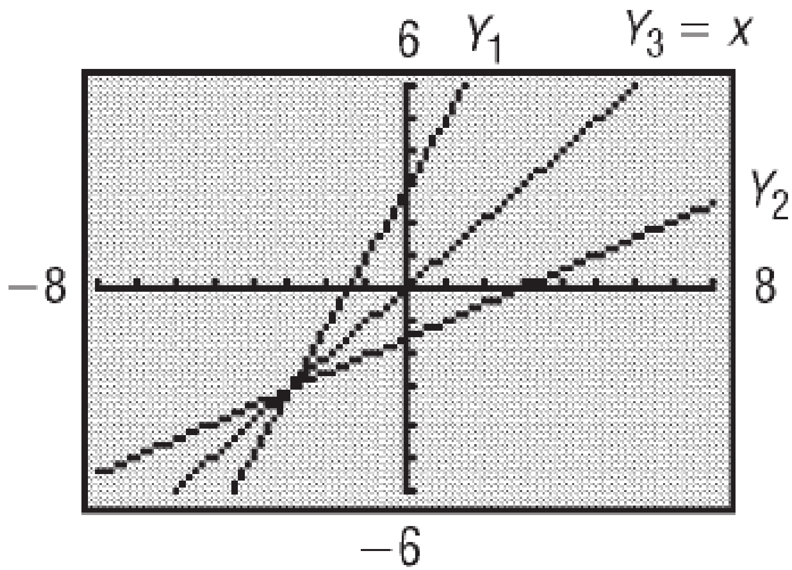


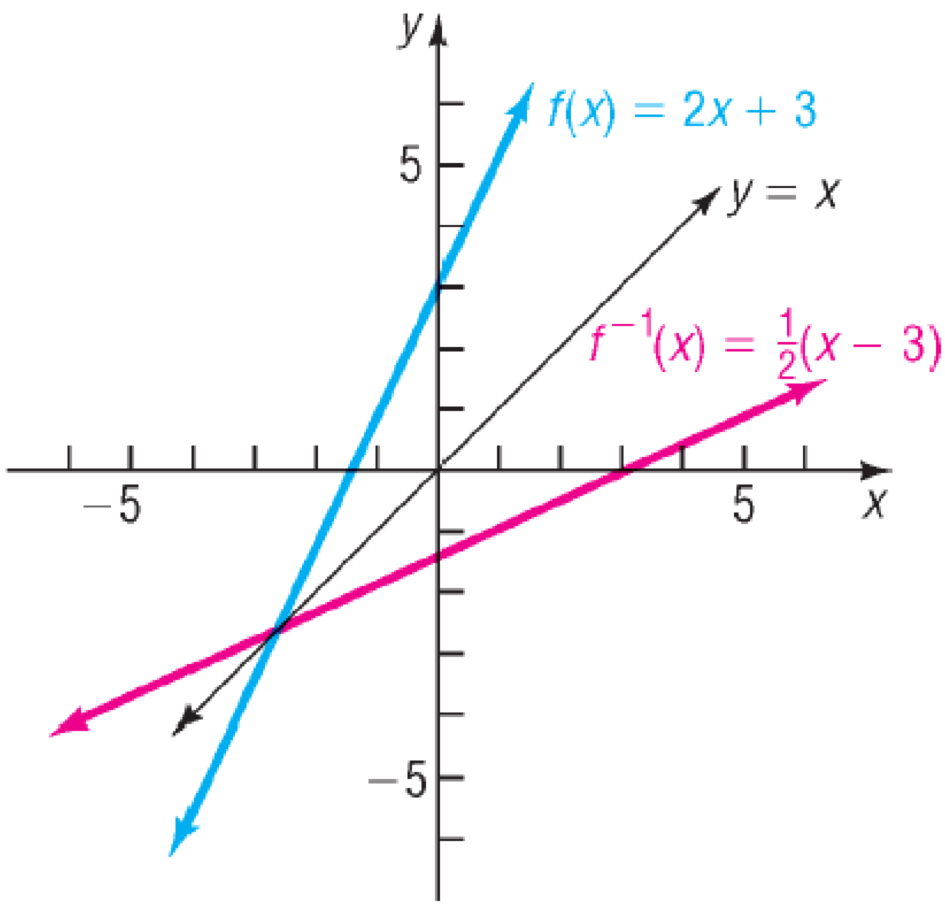
OBJECTIVE 4

4 **Find the Inverse of a Function Defined by an Equation**

EXAMPLE Finding the Inverse Function

Find the inverse of $f(x) = 2x + 3$. Also find the domain and range of f and f^{-1}. Graph f and f^{-1} on the same coordinate axes.





Procedure for Finding the Inverse of a One-to-One Function

STEP 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

STEP 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}

$$y = f^{-1}(x)$$

STEP 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE Finding the Inverse Function

The function

$$f(x) = \frac{2x + 1}{x - 1}, \qquad x \neq 1$$

is one-to-one. Find its inverse and check the result.

Step 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

Step 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}

$$y = f^{-1}(x)$$

Step 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

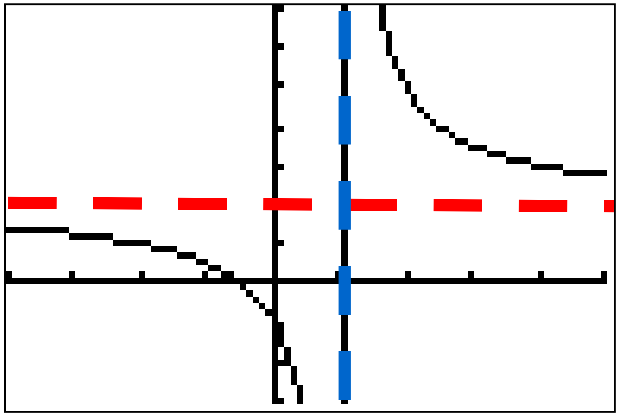
Exploration

In Example 9, we found that, if $f(x) = \dfrac{2x+1}{x-1}$, then $f^{-1}(x) = \dfrac{x+1}{x-2}$.

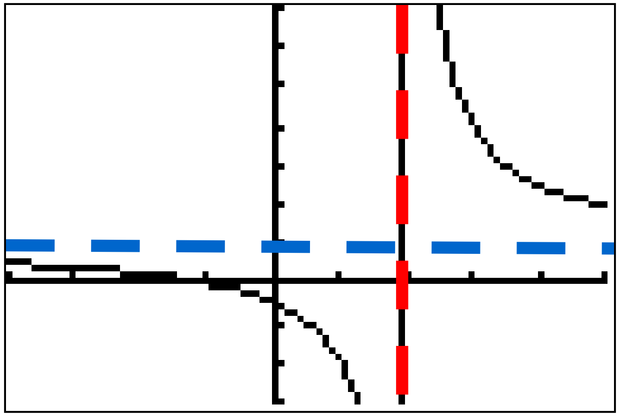
Compare the vertical and horizontal asymptotes of f and f^{-1}.

. What did you find? Are you surprised?

$f(x) = \dfrac{2x+1}{x-1}$



$f^{-1}(x) = \dfrac{x+1}{x-2}$



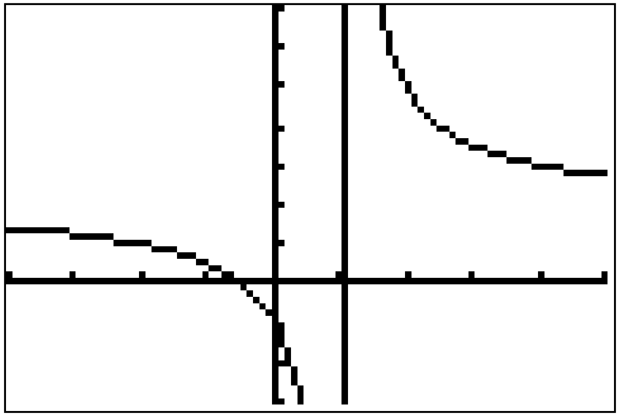
EXAMPLE

Finding the Range of a Function

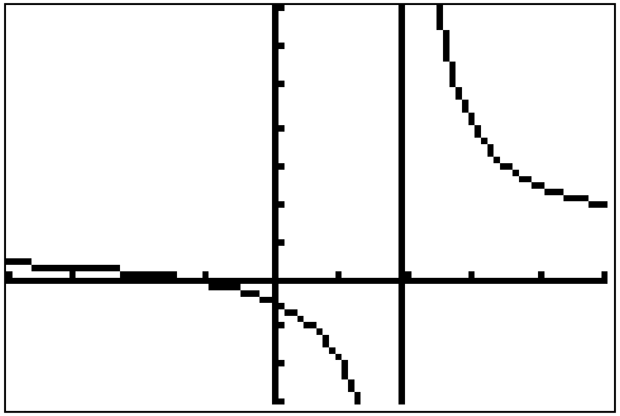
Find the domain and range of

$$f(x) = \frac{2x + 1}{x - 1}$$

$$f(x) = \frac{2x + 1}{x - 1}$$



$$f^{-1}(x) = \frac{x + 1}{x - 2}$$



EXAMPLE

Finding the Inverse of a Domain-restricted Function

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$.

Step 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

Step 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}

$$y = f^{-1}(x)$$

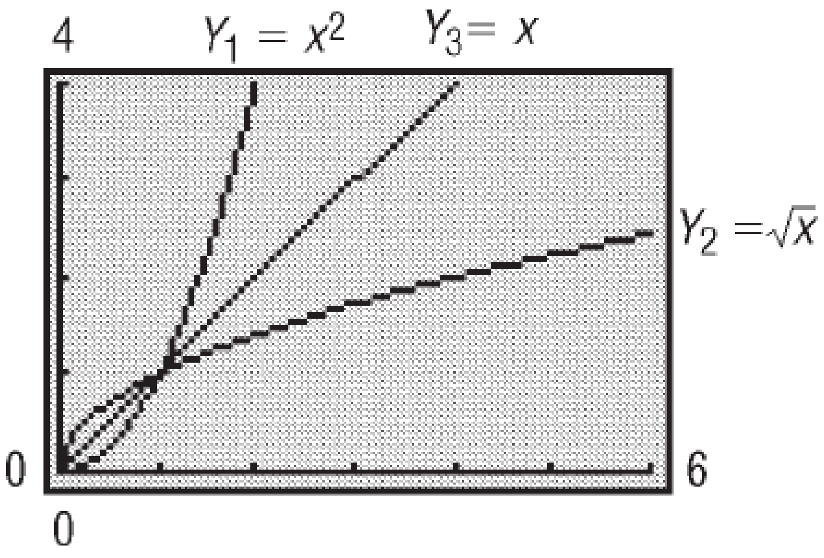
Step 3: Check the result by showing that

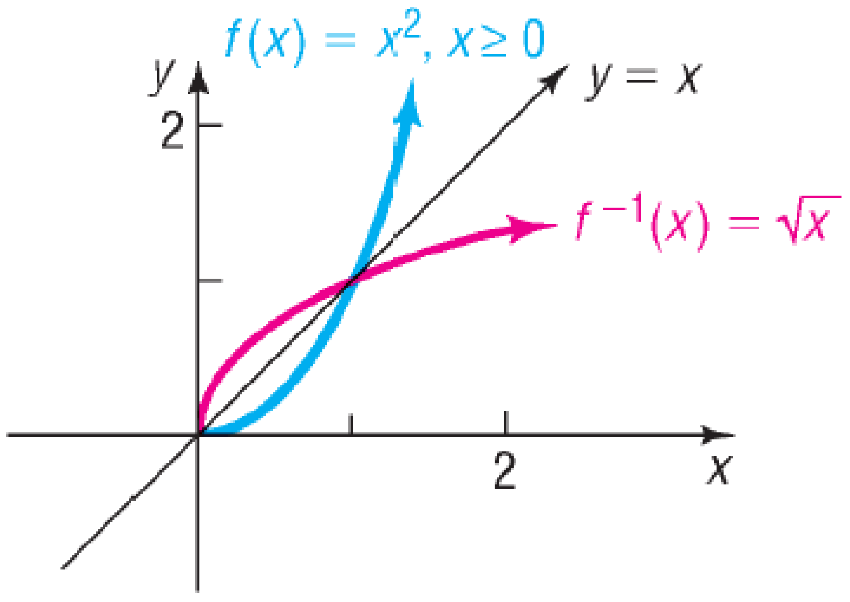
$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE

Finding the Inverse of a Domain-restricted Function

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$.





Summary

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.
5. To find the range of a one-to-one function f, find the domain of the inverse function f^{-1}.