Section 4.3

Exponential Functions

OBJECTIVE 1

1 Evaluate Exponential Functions

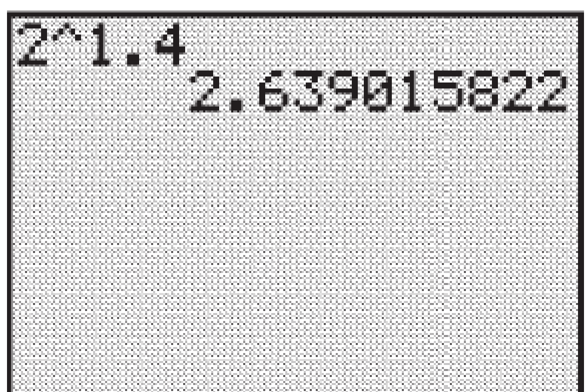
EXAMPLE

Using a Calculator to Evaluate Powers of 2

Using a calculator, evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Most calculators have an x^y key or a caret key $\wedge$ for working with exponents. To evaluate expressions of the form a^x, enter the base a, then press the x^y key (or the $\wedge$ key), enter the exponent x, and press $=$ (or enter).



Theorem

Laws of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s \cdot a^t = a^{s+t} \qquad (a^s)^t = a^{st} \qquad (ab)^s = a^s \cdot b^s$$

$$1^s = 1 \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1$$

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive real number ($a > 0$) and $a \neq 1$.

The domain of f is the set of all real numbers.

Exploration

(a) Evaluate $f(x) = 2^x$ at $x = -2, -1, 0, 1, 2,$ and 3.
(b) Evaluate $g(x) = 3x + 2$ at $x = -2, -1, 0, 1, 2,$ and 3.
(c) Comment on the pattern that exists in the values of f and g.

x	$f(x) = 2^x$
-2	$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x	$g(x) = 3x + 2$
-2	$g(-2) = 3(-2) + 2 = -4$
-1	-1
0	2
1	5
2	8
3	11

Theorem

For an exponential function $f(x) = a^x$, $a > 0, a \neq 1$, if x is any real number, then

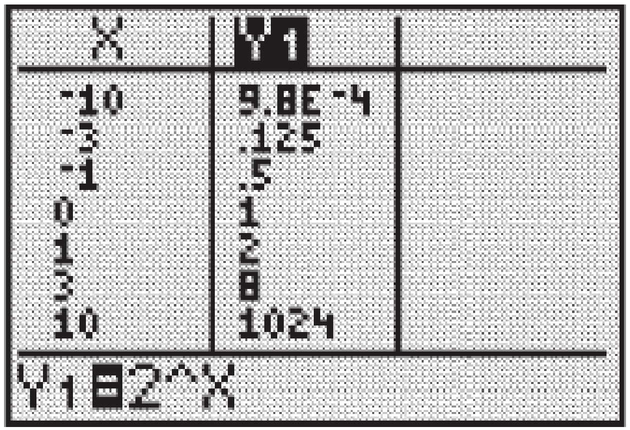
$$\frac{f(x+1)}{f(x)} = a$$

OBJECTIVE 2

2 Graph Exponential Functions

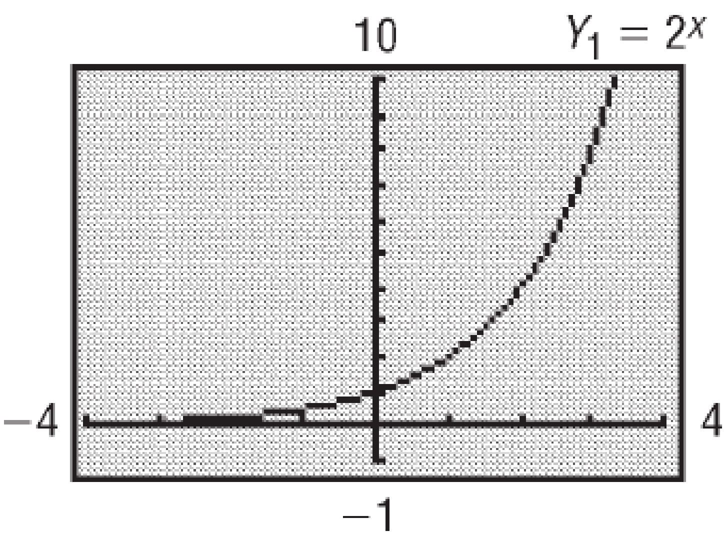
EXAMPLE

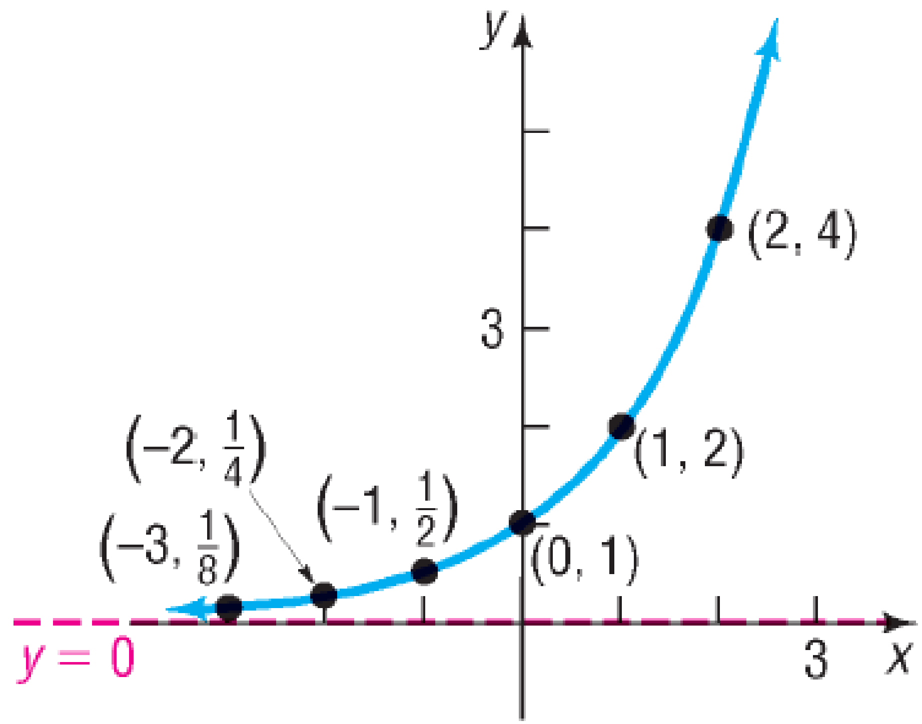
Graph the exponential function: $f(x) = 2^x$

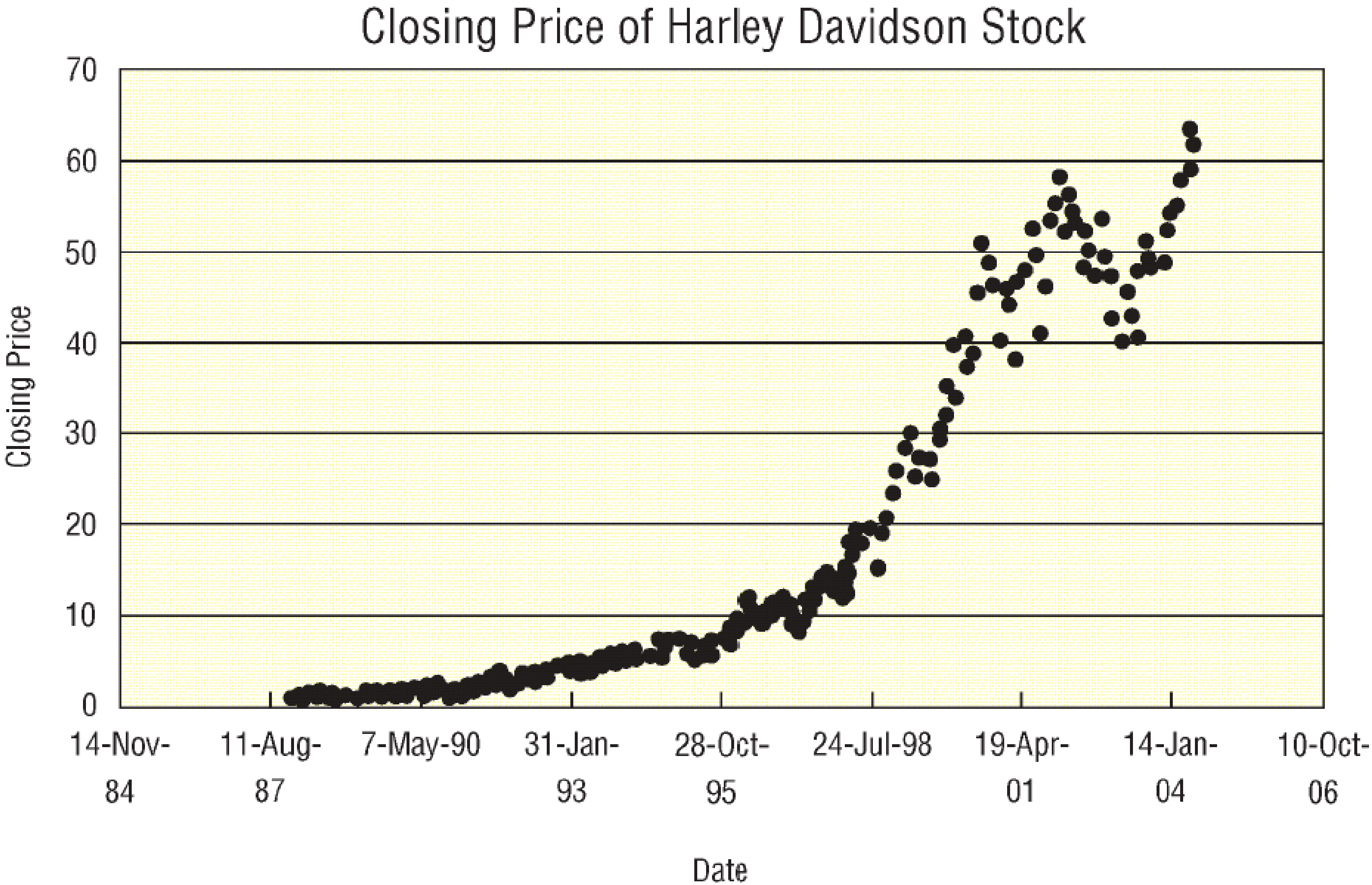


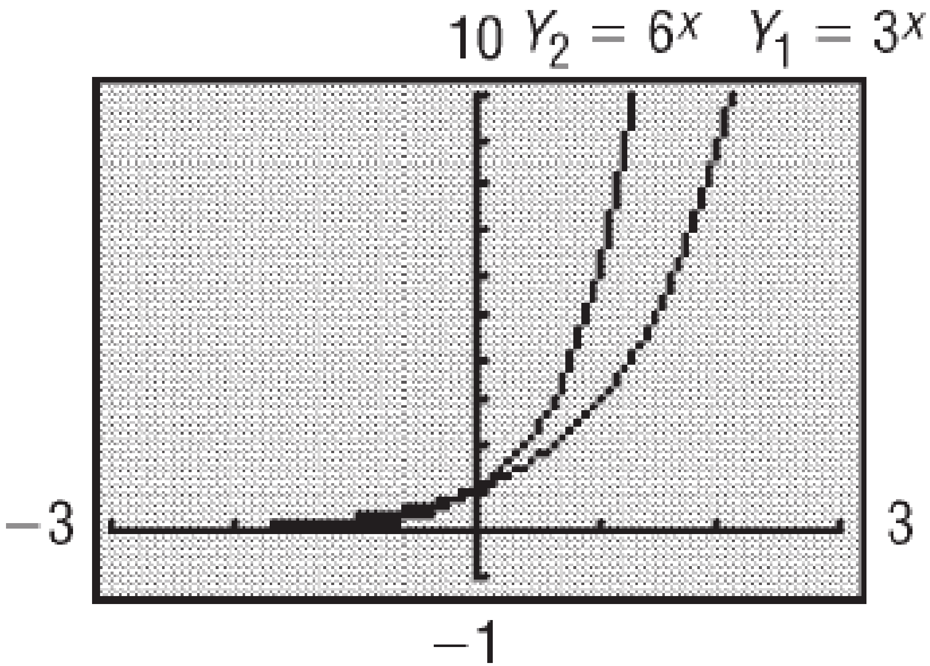
X	Y1
-10	9.8E-4
-3	.125
-1	.5
0	1
1	2
3	8
10	1024

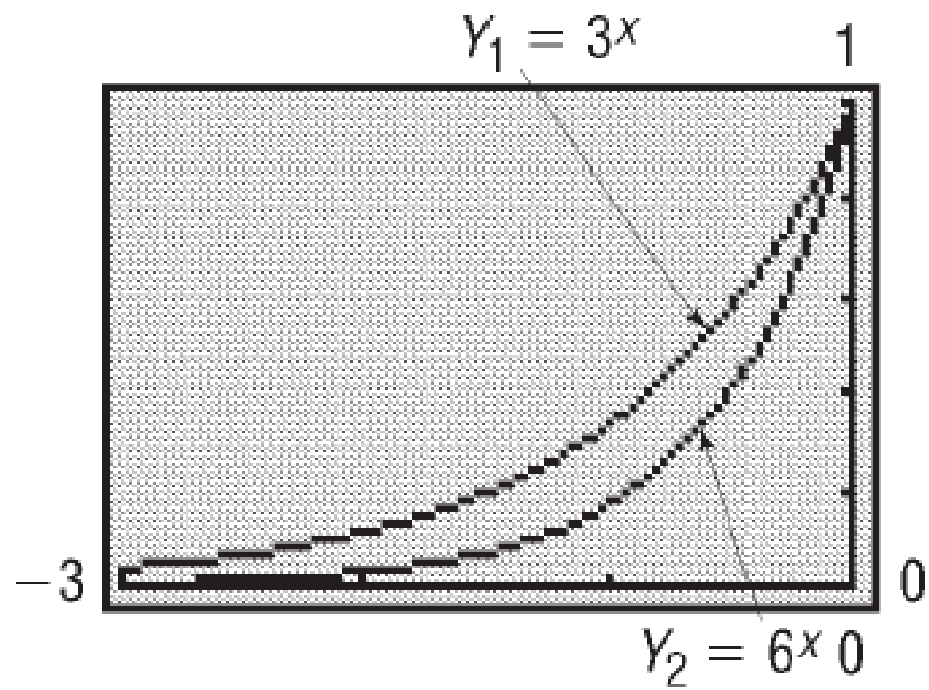
Y1=2^X





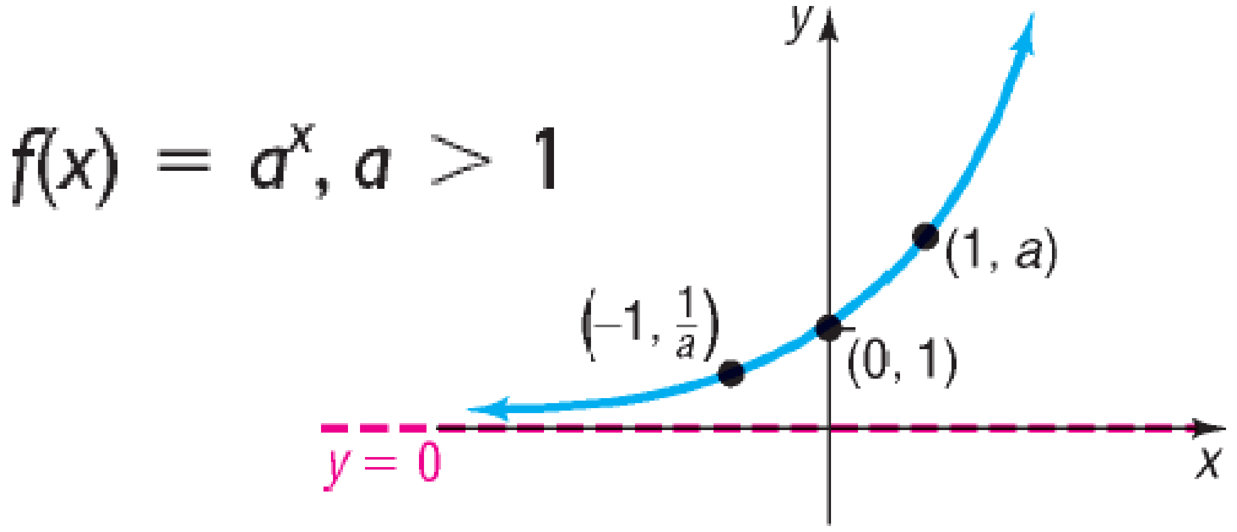

Closing Price of Harley Davidson Stock
Closing Price
70
60
50
40
30
20
10
0
14-Nov-84
11-Aug-87
7-May-90
31-Jan-93
28-Oct-95
24-Jul-98
19-Apr-01
14-Jan-04
10-Oct-06
Date


10
$Y_2 = 6^x$
$Y_1 = 3^x$
−3
3
−1


$Y_1 = 3^x$
1
−3
0
$Y_2 = 6^x$
0

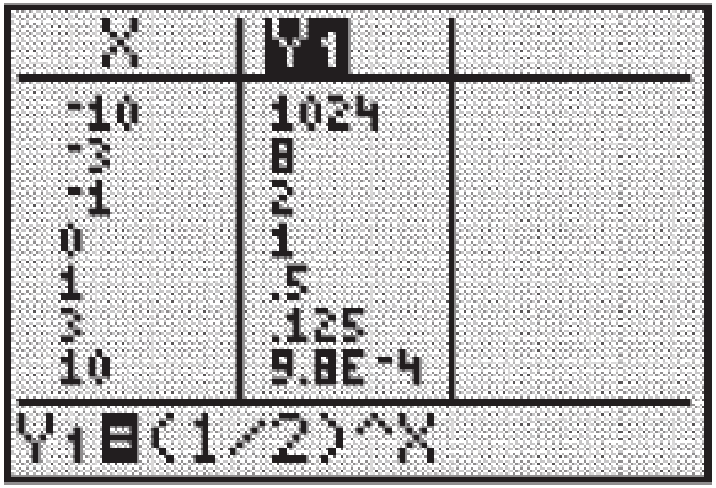
Properties of the Exponential Function $f(x) = a^x$, $a > 1$

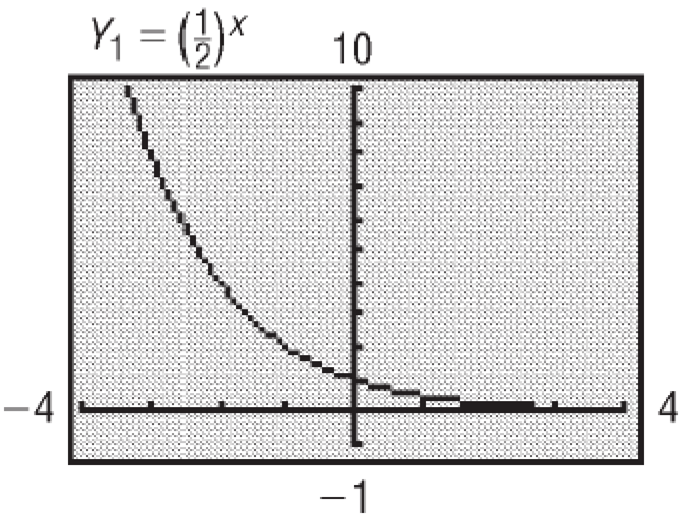
1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to -\infty$.
4. $f(x) = a^x$, $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps.

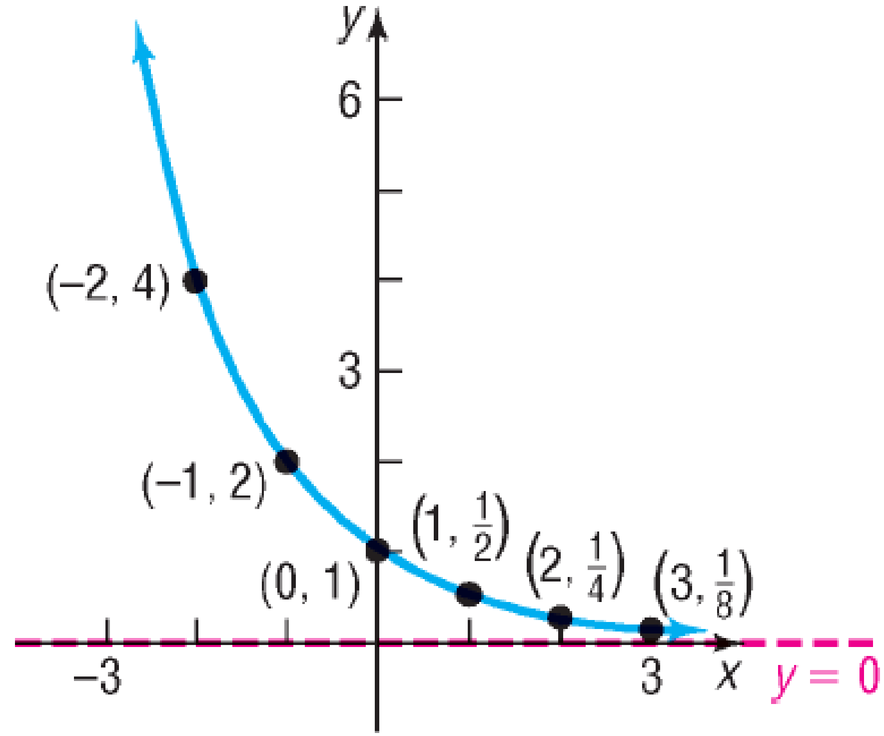


EXAMPLE Graphing an Exponential Function

Graph the exponential function: $f(x) = \left(\frac{1}{2}\right)^x$







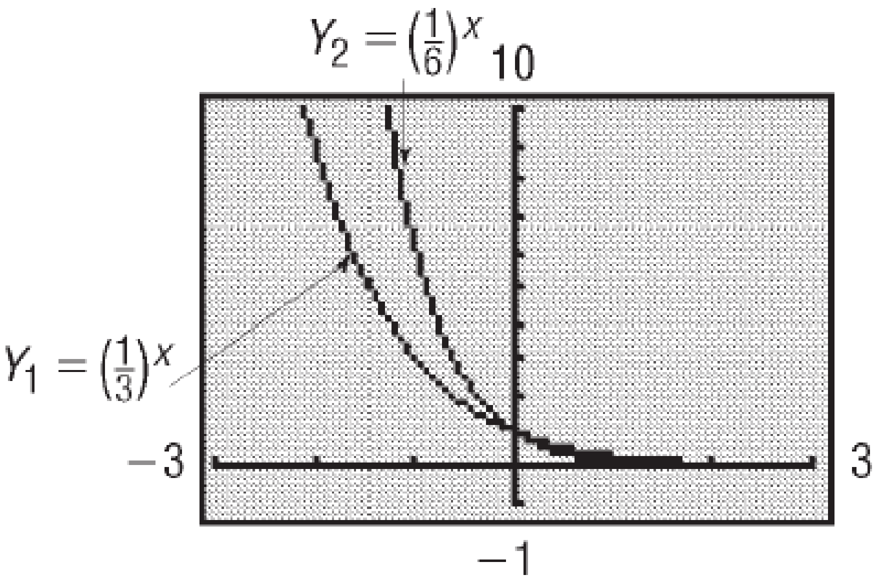
Seeing the Concept

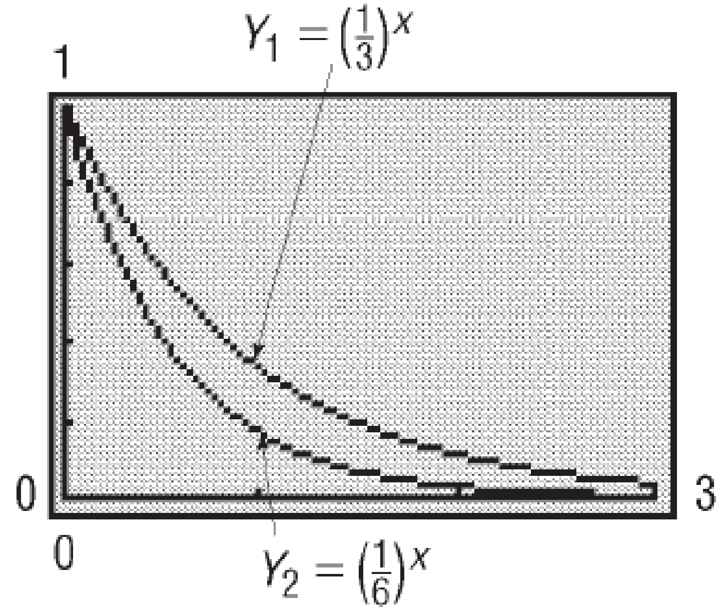
Using a graphing utility, simultaneously graph:

(a) $Y_1 = 3^x,\ Y_2 = \left(\dfrac{1}{3}\right)^x$

(b) $Y_1 = 6^x,\ Y_2 = \left(\dfrac{1}{6}\right)^x$

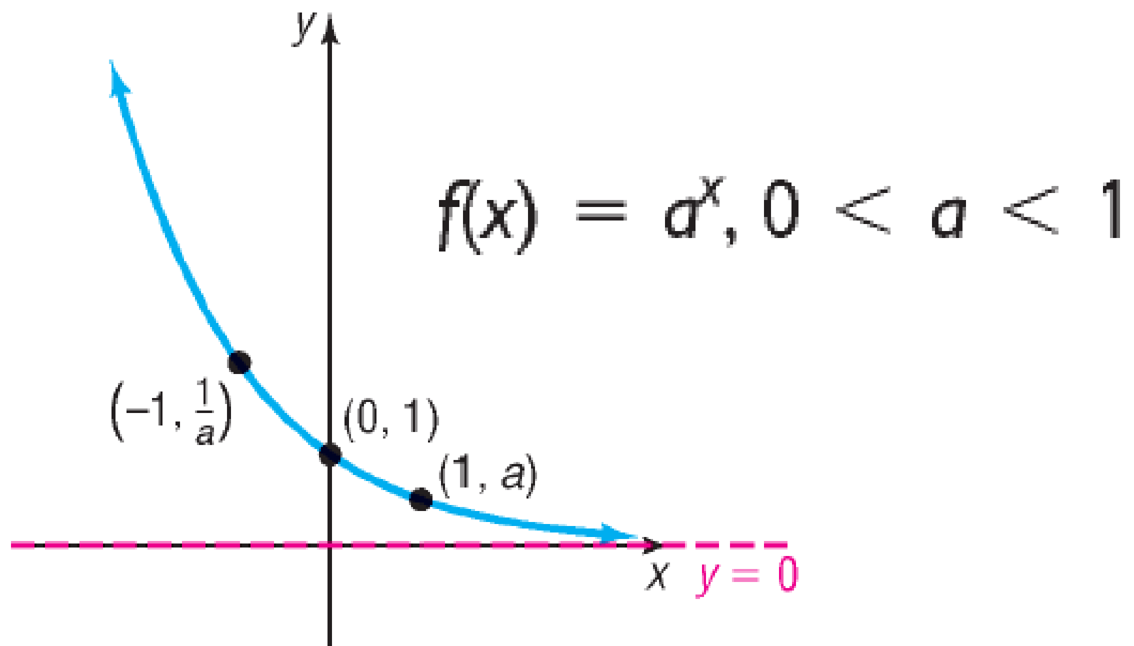
Conclude that the graph of $Y_2 = \left(\dfrac{1}{a}\right)^x$, for $a > 0$, is the reflection about the y-axis of the graph of $Y_1 = a^x$.





Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

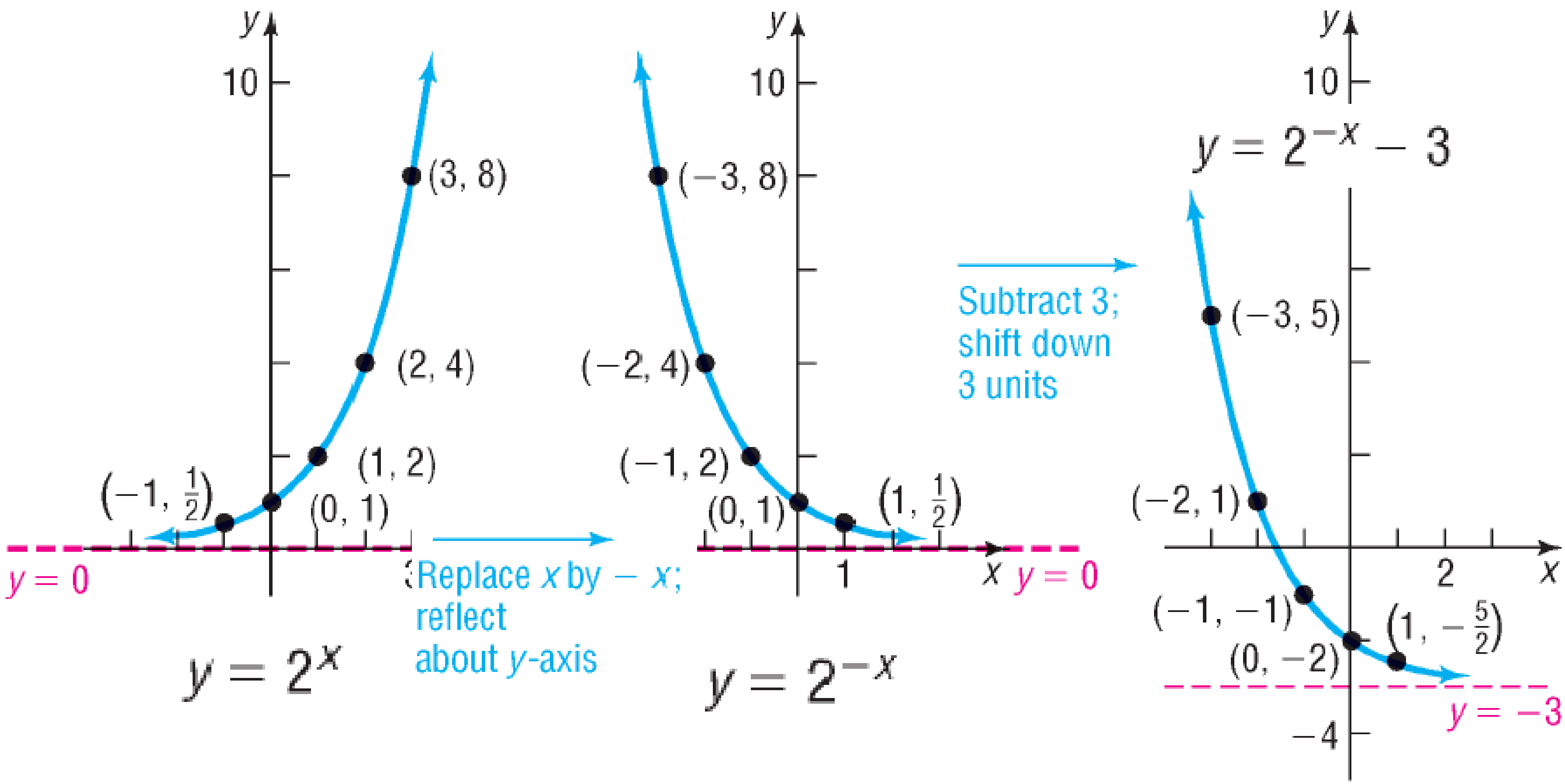
1. The domain is the set of all real numbers; the range is the set of positive real numbers.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis ($y = 0$) is a horizontal asymptote as $x \to \infty$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps.



EXAMPLE

Graphing Exponential Functions Using Transformations

Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, and horizontal asymptote of f.



OBJECTIVE 3

3 Define the Number e

The **number *e*** is defined as the number that the expression

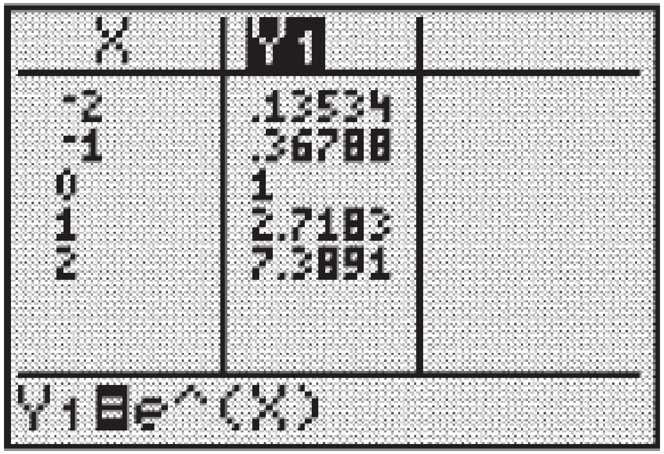
$$\left(1 + \frac{1}{n}\right)^n$$

approaches as $n \to \infty$.

In calculus, this is expressed using limit notation as

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

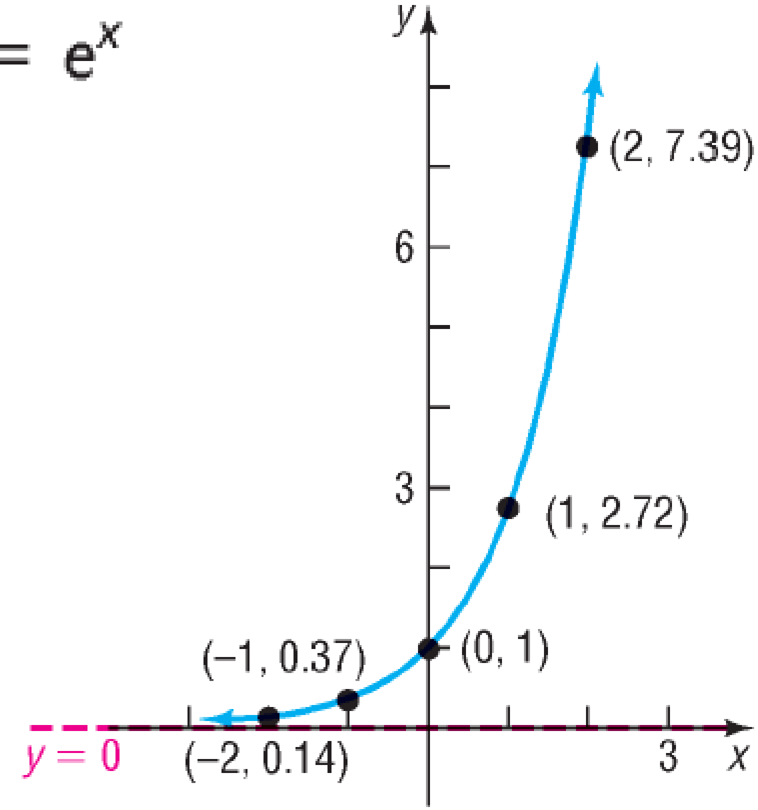
n	$\frac{1}{n}$	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
1,000,000,000	10^{-9}	$1+10^{-9}$	2.718281827

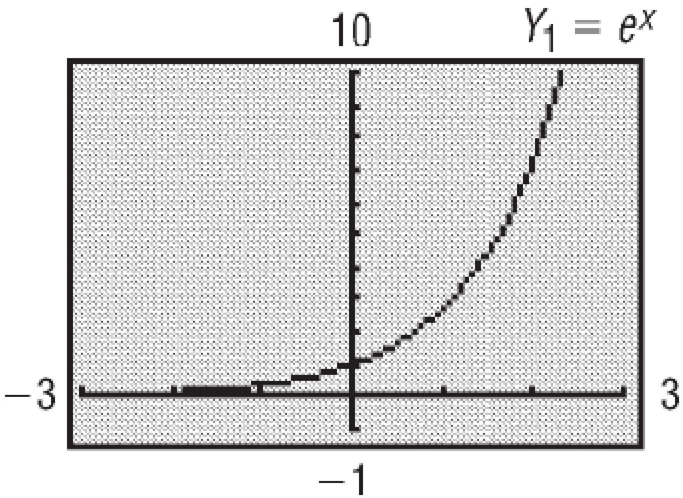


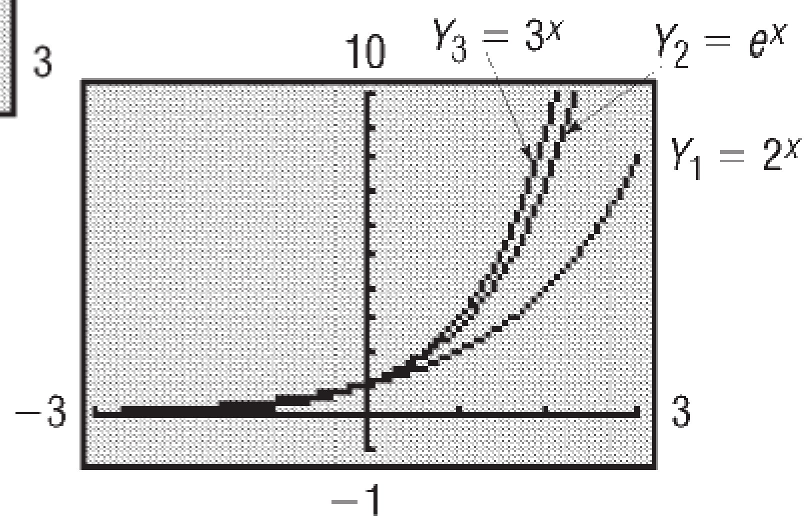
X	Y1
-2	.13534
-1	.36788
0	1
1	2.7183
2	7.3891

Y1 = e^(X)

$y = e^x$



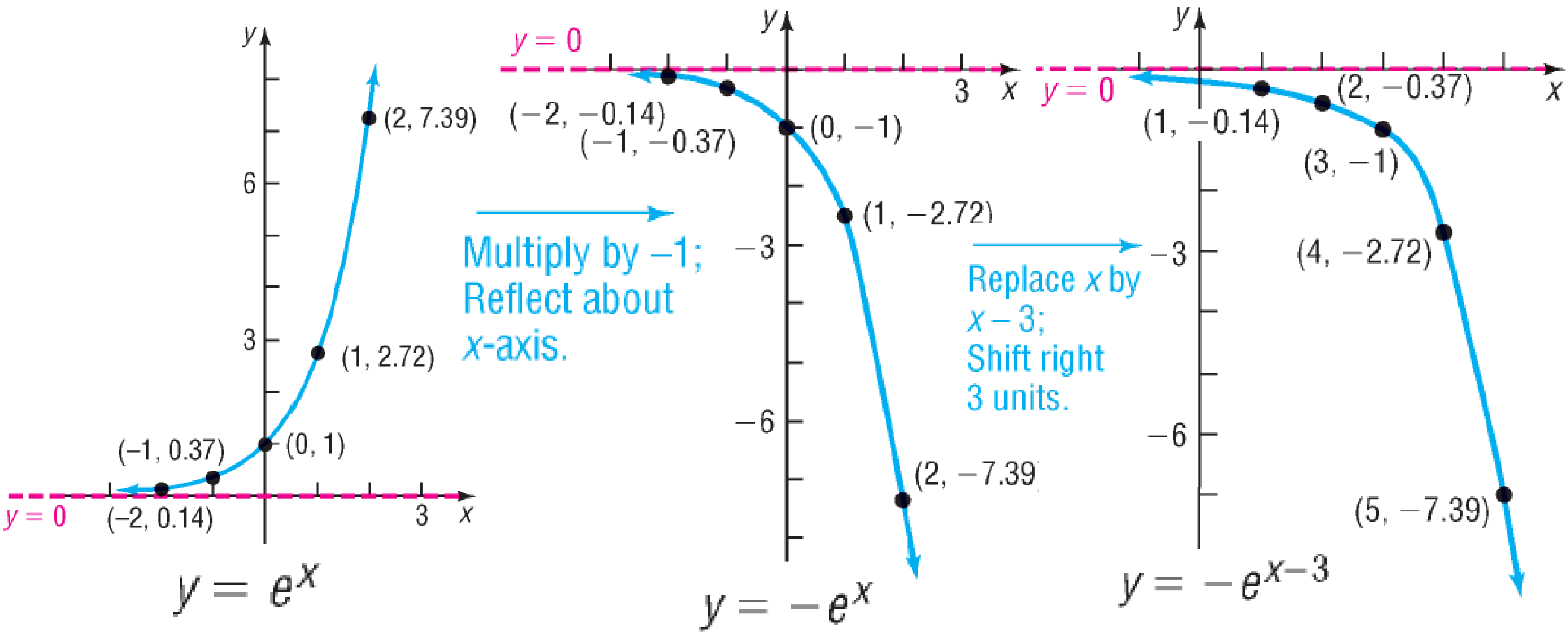




EXAMPLE

Graphing Exponential Functions Using Transformations

Graph $f(x) = -e^{x-3}$ and determine the domain, the range, and horizontal asymptote of f.



OBJECTIVE 4

4 **Solve Exponential Equations**

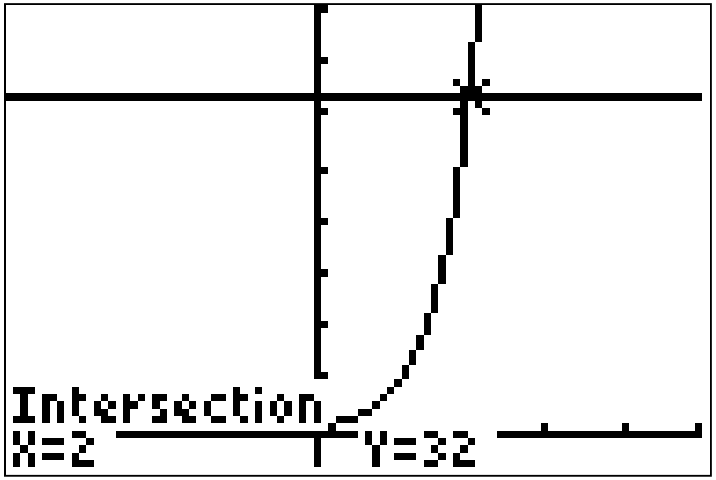
$$\text{If} \quad a^u = a^v, \qquad \text{then} \quad u = v$$

EXAMPLE

Solving an Exponential Equation

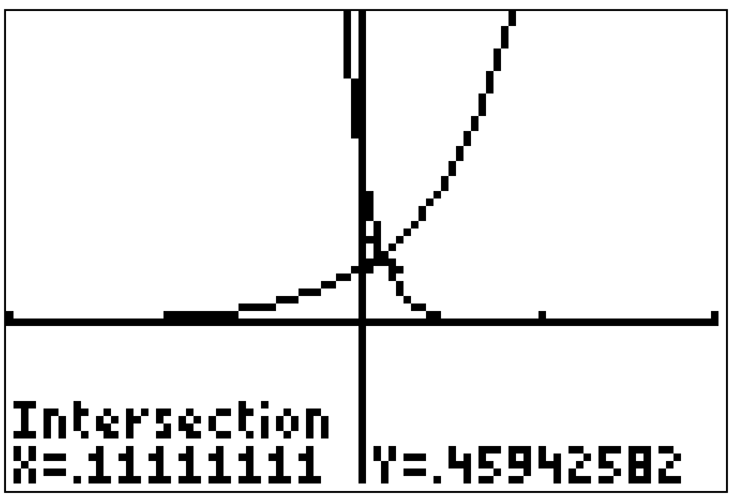
Solve: $2^{3x-1} = 32$

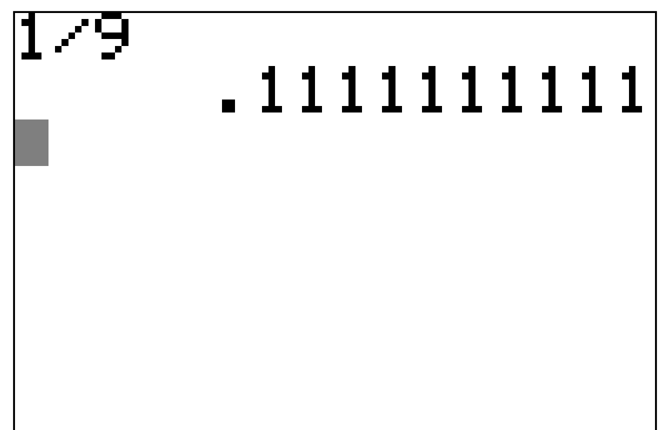
If $a^u = a^v$, then $u = v$



EXAMPLE Solving an Exponential Equation

$$\text{Solve: } e^{2x-1} = \frac{1}{e^{3x}} \cdot \left(e^{-x}\right)^4$$





EXAMPLE Exponential Probability

Between 9:00 PM and 10:00 PM cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

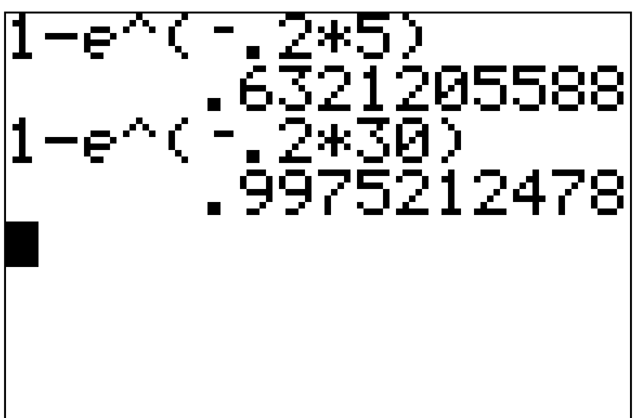
$$F(t) = 1 - e^{-0.2t}$$

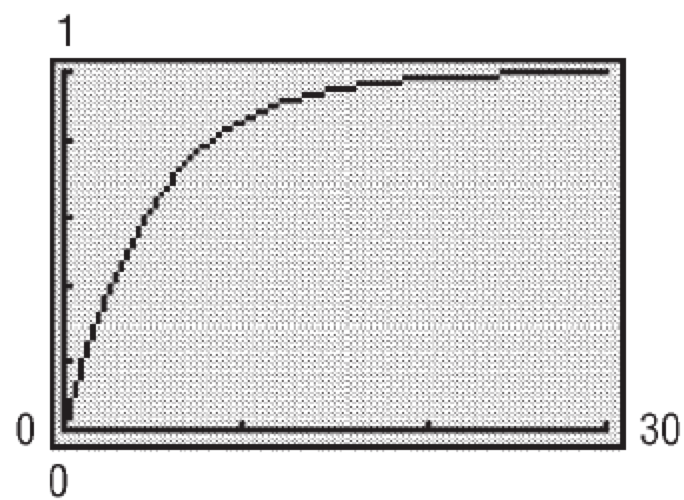
(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

(c) Graph F using your graphing utility.

(d) What value does F approach as t becomes unbounded in the positive direction?

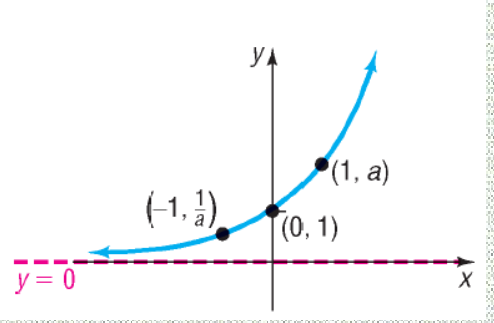




Summary

Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$



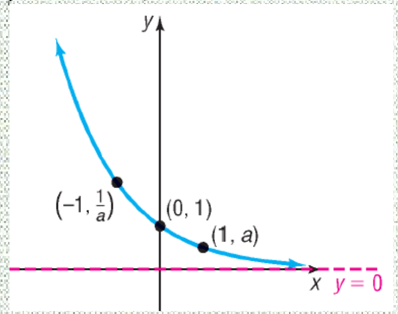
Domain: the interval $(-\infty, \infty)$; Range: the interval $(0, \infty)$

x-intercepts: none; y-intercept: 1

Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$

Increasing; one-to-one; smooth; continuous

$f(x) = a^x, \quad 0 < a < 1$



Domain: the interval $(-\infty, \infty)$; Range: the interval $(0, \infty)$.

x-intercepts: none; y-intercept: 1

Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$

Decreasing; one-to-one; smooth; continuous

If $a^u = a^v$, then $u = v$.