Section 4.4

Logarithmic Functions

The **logarithmic function to the base** $\boldsymbol{a}$, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

EXAMPLE

Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$.

For example, $4 = \log_3 81$ is equivalent to $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$.

For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

OBJECTIVE 1

Change Exponential Expressions to Logarithmic Expressions and Logarithmic Expressions to Exponential Expressions

EXAMPLE

Changing Exponential Expressions to Logarithmic Expressions

Change each exponential expression to an equivalent expression involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

EXAMPLE

Changing Logarithmic Expressions to Exponential Expressions

Change each logarithmic expression to an equivalent expression involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

OBJECTIVE 2

2 Evaluate Logarithmic Expressions

EXAMPLE

Finding the Exact Value of a Logarithmic Expression

$$(a)\ \log_3 81 \qquad\qquad (b)\ \log_2 \frac{1}{8}$$

OBJECTIVE 3

3 Determine the Domain of a Logarithmic Function

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

EXAMPLE

Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

$(a)\ f(x)=\log_3(x-2)$

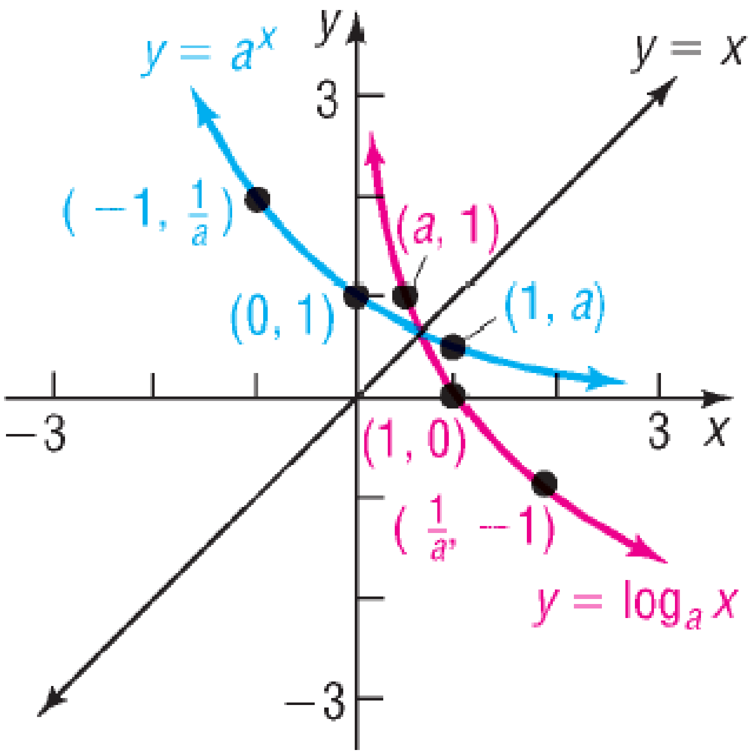
$(b)\ F(x)=\log_2\left(\frac{x+3}{x-1}\right)$

$(c)\ h(x)=\log_2|x-1|$

$(d)\ g(x)=\log_{\frac{1}{2}} x^2$

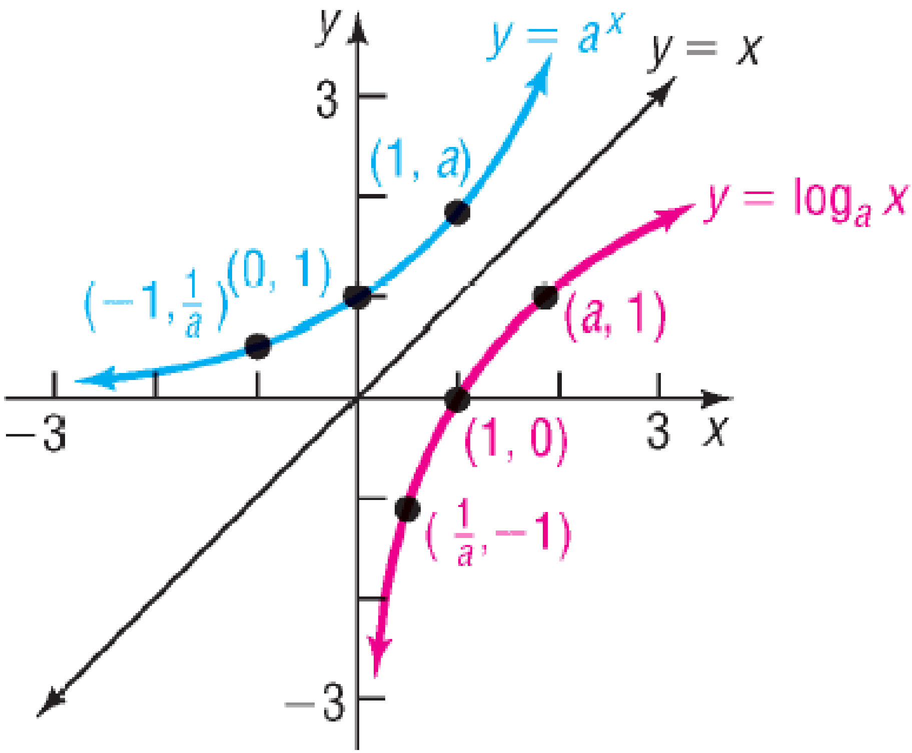
OBJECTIVE 4

4 Graph Logarithmic Functions



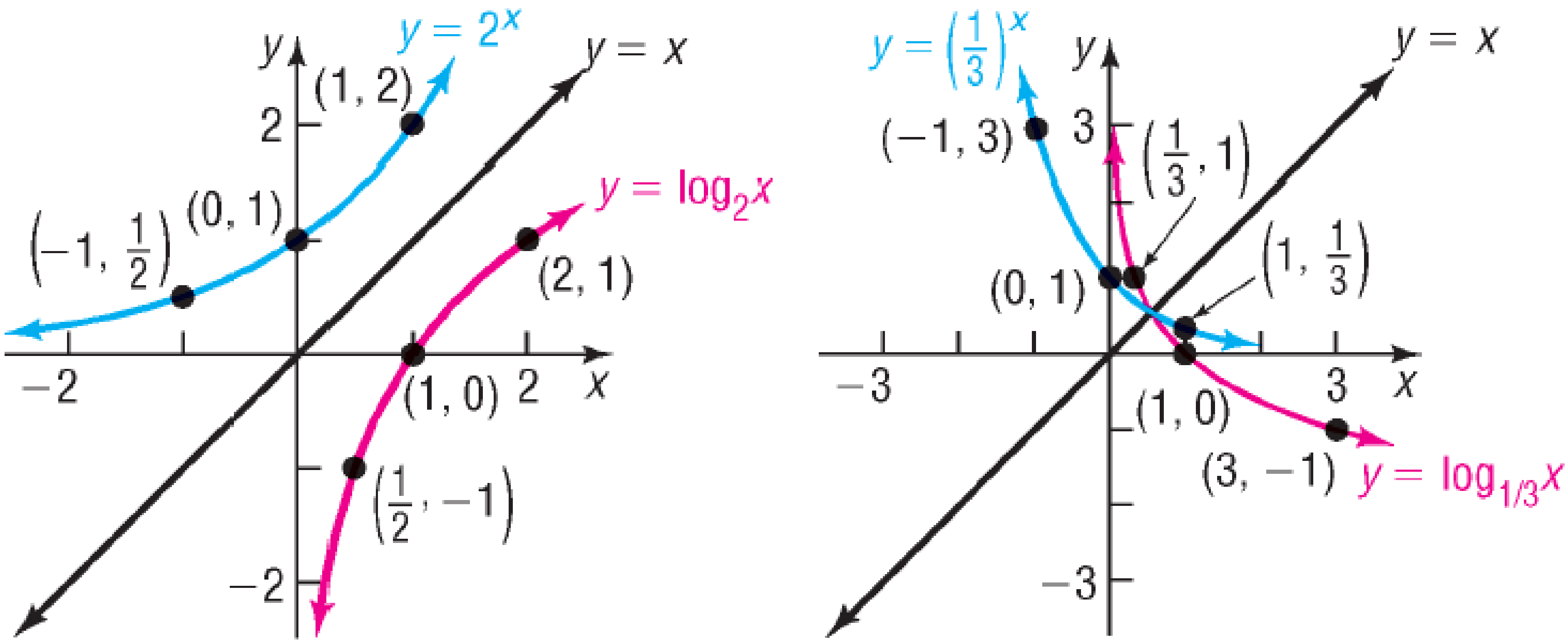
y
y = a^x
y = x
3
(−1, 1/a)
(a, 1)
(0, 1)
(1, a)
−3
3
x
(1, 0)
(1/a, −1)
y = log_a x
−3

(a) $0 < a < 1$



y
y = a^x
y = x
3
(1, a)
y = log_a x
(−1, 1/a)
(0, 1)
(a, 1)
−3
3
x
(1, 0)
(1/a, −1)
−3

(b) $a > 1$



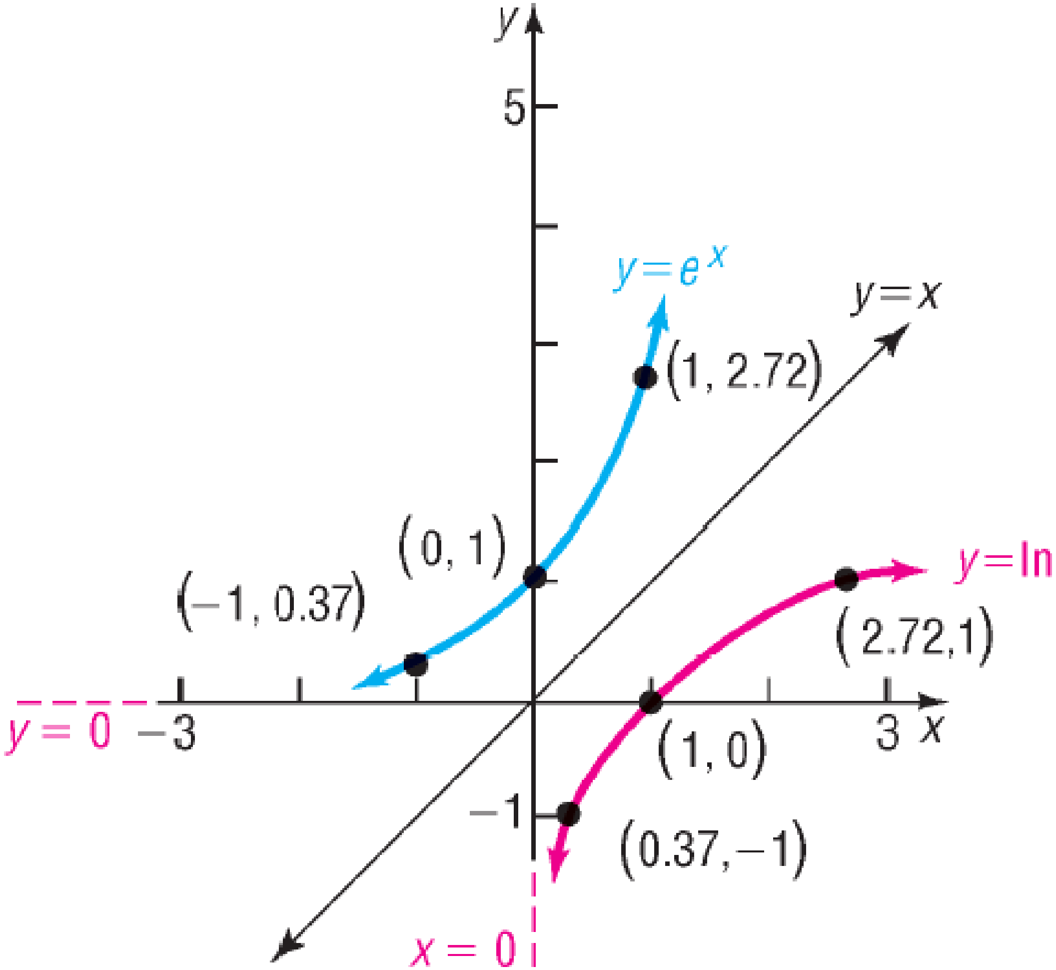
y = 2^x
y = x
y
(1, 2)
2
(0, 1)
(−1, 1/2)
y = log_2 x
(2, 1)
−2
(1, 0)
2
x
(1/2, −1)
−2
y = (1/3)^x
y
y = x
(−1, 3)
3
(1/3, 1)
(1, 1/3)
(0, 1)
−3
(1, 0)
3
x
(3, −1)
y = log_1/3 x
−3

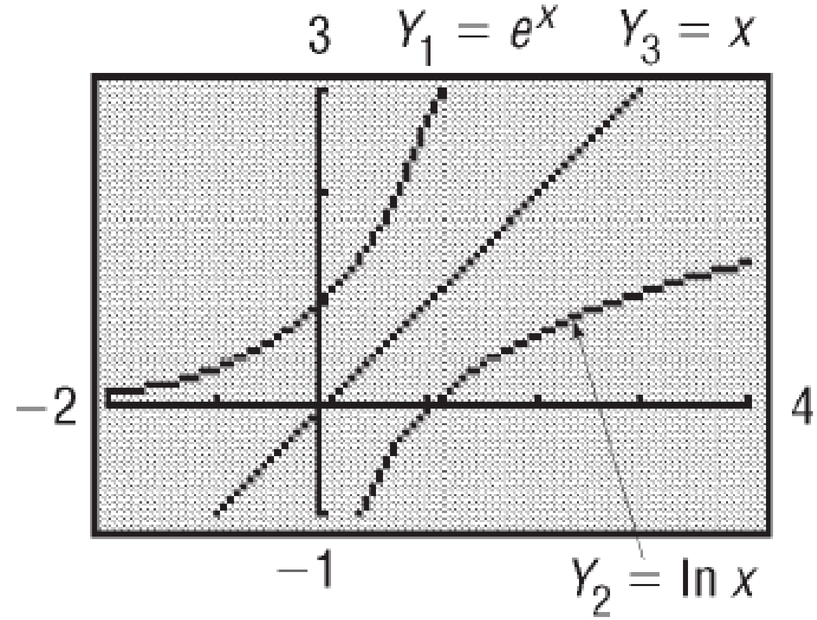
Properties of the Logarithmic Function $f(x) = \log_a x$

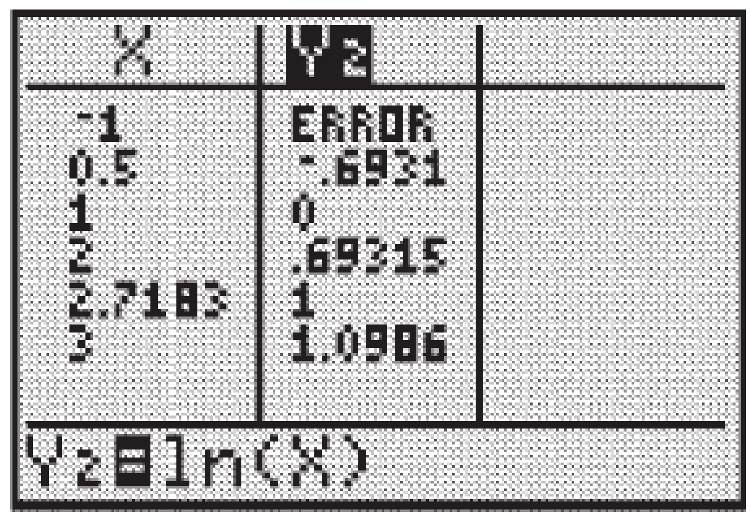
1. The domain is the set of positive real numbers; the range is the set of all real numbers.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph is smooth and continuous, with no corners or gaps.

Natural Logarithm Function

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$







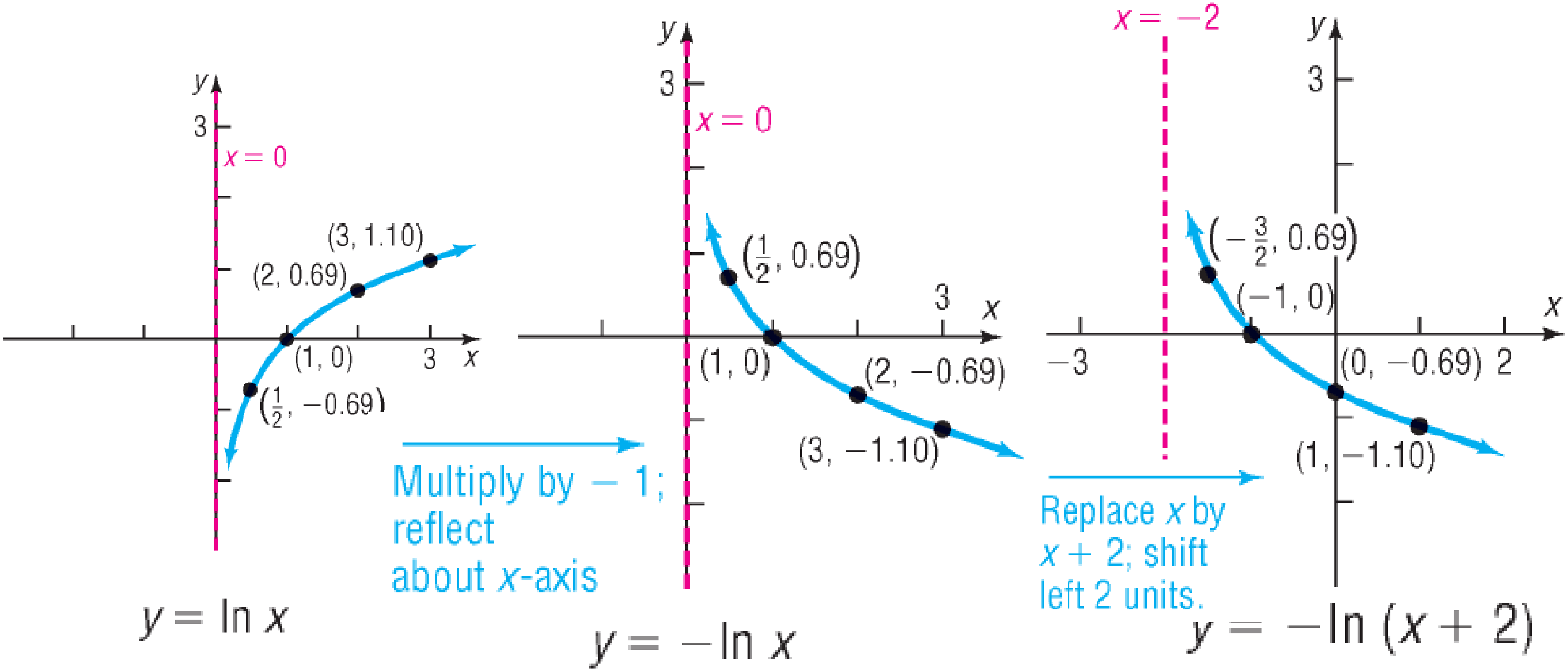
X	Y2	
-1	ERROR	
0.5	-.6931	
1	0	
2	.69315	
2.7183	1	
3	1.0986	

Y2=ln(X)

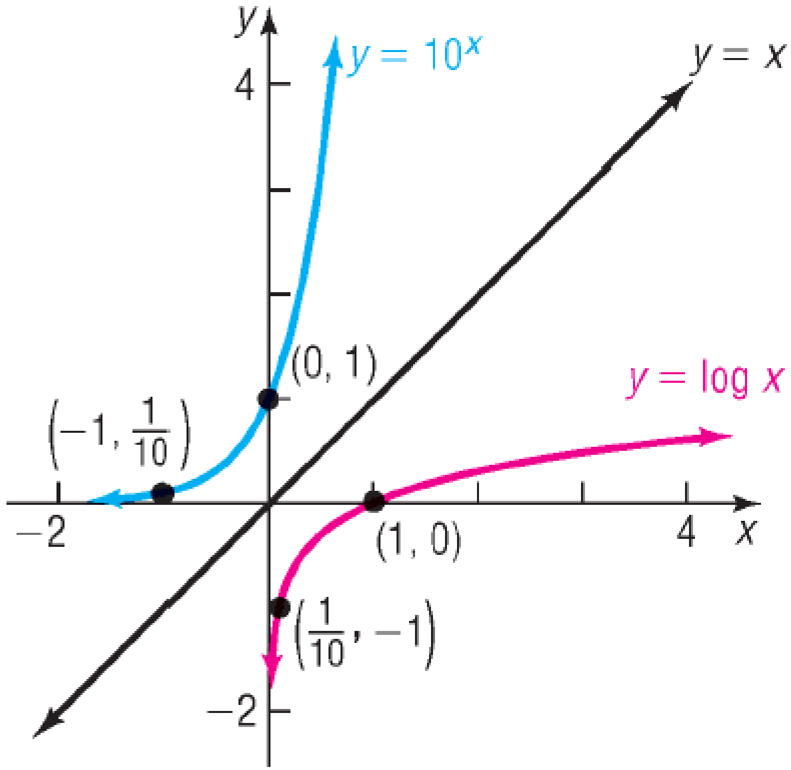
EXAMPLE

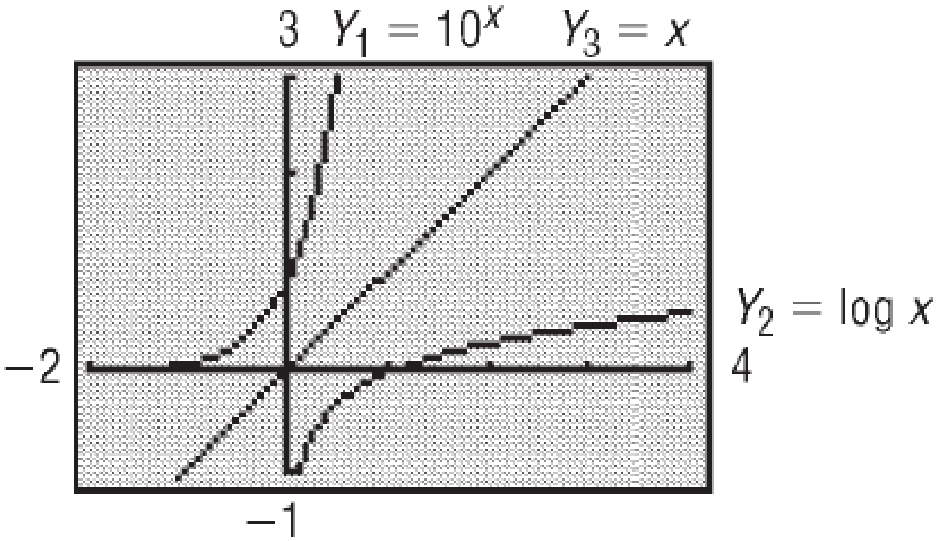
Graphing Logarithmic Functions Using Transformations

Graph $f(x) = -\ln(x + 2)$ by starting with the graph of $y = \ln x$. Determine the domain, range, and vertical asymptote.



$$y = \log x \quad \text{if and only if} \quad x = 10^y$$

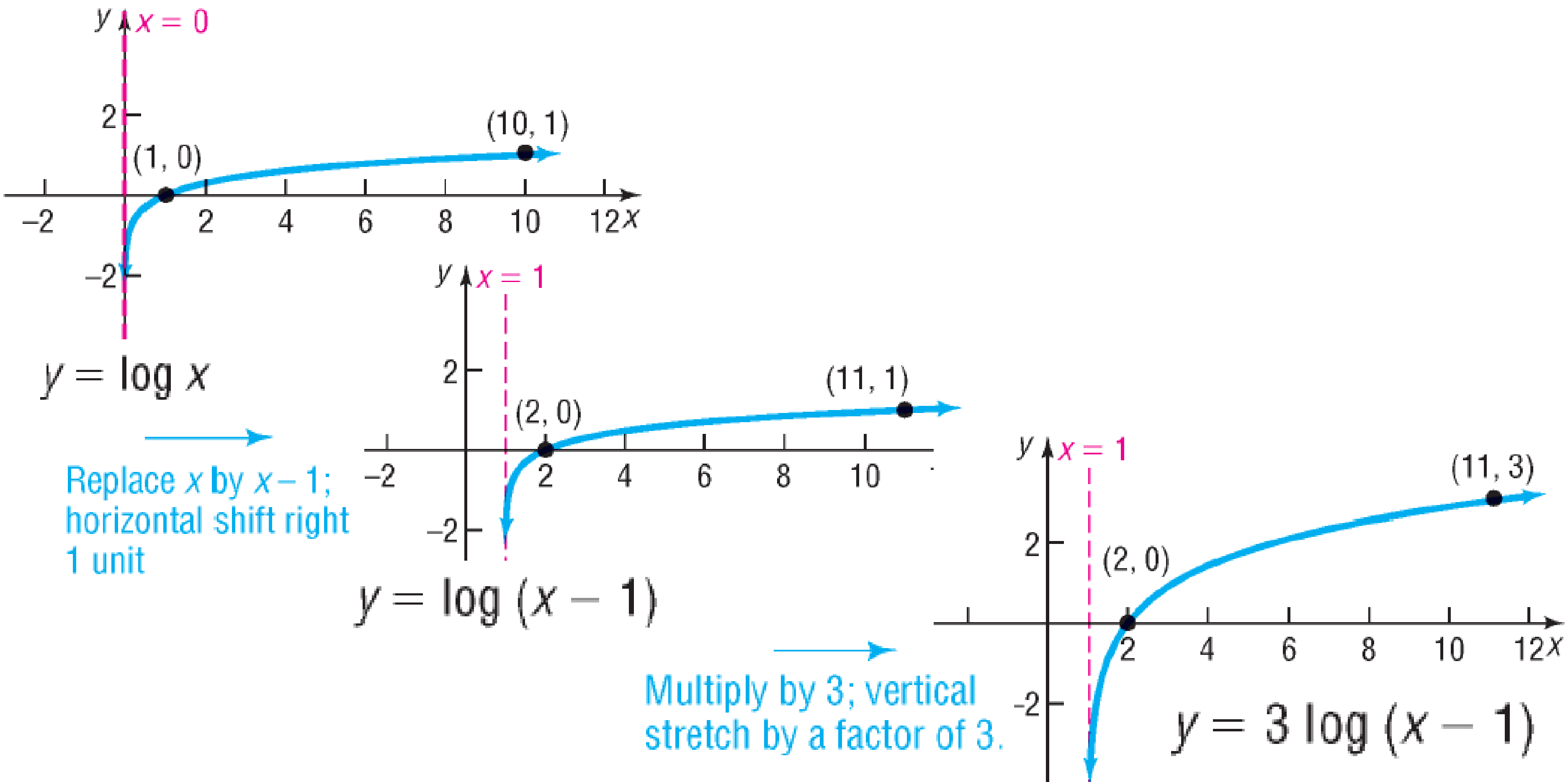




EXAMPLE

Graphing Logarithmic Functions Using Transformations

Graph $f(x) = 3\log(x - 1)$. Determine the domain, range, and vertical asymptote of f.



OBJECTIVE 5

5 Solve Logarithmic Equations

EXAMPLE

Solving a Logarithmic Equation

Solve: $(a)\ \log_2(2x+1)=3$ $\quad (b)\ \log_x 343 = 3$

EXAMPLE

Using Logarithms to Solve Exponential Equations

Solve: $2e^{3x} = 6$

EXAMPLE Alcohol and Driving

The concentration of alcohol in a person's blood is measurable. Recent medical research suggests that the risk R (given as a percent) of having an accident while driving a car can be modeled by the equation

$$R = 6e^{kx}$$

where x is the variable concentration of alcohol in the blood and k is a constant.

(a) Suppose that a concentration of alcohol in the blood of 0.04 results in a 10% risk ($R = 10$) of an accident. Find the constant k in the equation. Graph $R = 6e^{kx}$ using this value of k.

(b) Using this value of k, what is the risk if the concentration is 0.17?

(c) Using the same value of k, what concentration of alcohol corresponds to a risk of 100%?

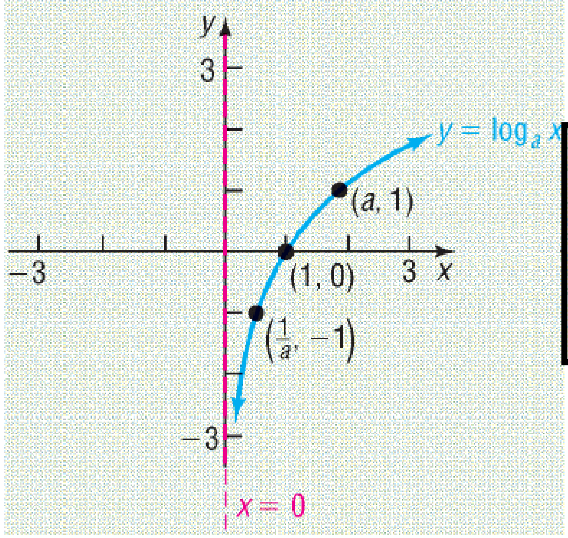
(d) If the law asserts that anyone with a risk of having an accident of 20% or more should not have driving privileges, at what concentration of alcohol in the blood should a driver be arrested and charged with a DUI (Driving Under the Influence)?

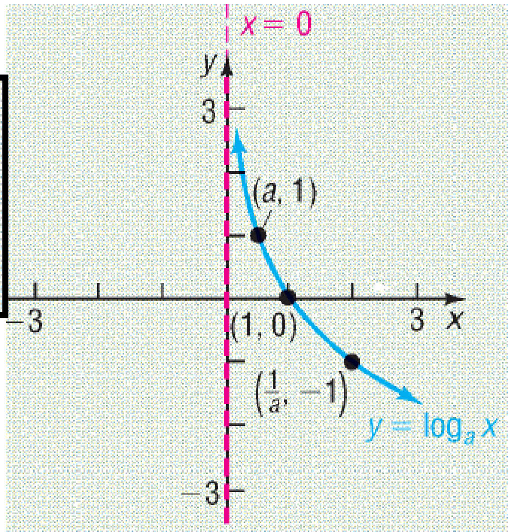
Summary

Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ — Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$

$(y = \log_a x$ means $x = a^y)$ — x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one





$f(x) = \log_a x, \quad 0 < a < 1$ — Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$

$(y = \log_a x$ means $x = a^y)$ — x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one