Section 4.7

Compound Interest

OBJECTIVE 1

1 Determine the Future Value of a Lump Sum of Money

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt$$

Interest charged according to formula (1) is called **simple interest**.

Annually:	Once per year
Semiannually:	Twice per year
Quarterly:	Four times per year
Monthly:	12 times per year
Daily:	365 times per year

EXAMPLE Computing Compound Interest

A credit union pays interest of 4% per annum compounded quarterly on a certain savings plan. If $2000 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year?

$$I = Prt$$

The new principal is $P + I$

The amount A after one compounding period is

$$A = P + I = P + P \cdot \left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)$$

After two compounding periods,

$$A = \underbrace{P \cdot \left(1 + \frac{r}{n}\right)}_{\text{New principal}} + \underbrace{P \cdot \left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)}_{\text{Interest on new principal}} = P \cdot \left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2$$

Theorem

Compound Interest Formula

The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

EXAMPLE

Comparing Investments Using Different Compounding Periods

Investing \$1000 at an annual rate of 10% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding ($n = 1$):

$$A = P \cdot (1 + r) = (\$1000)(1 + 0.10) = \$1100.00$$

Semiannual compounding ($n = 2$):

$$A = P \cdot \left(1 + \frac{r}{2}\right)^2 = (\$1000)(1 + 0.05)^2 = \$1102.50$$

Quarterly compounding ($n = 4$):

$$A = P \cdot \left(1 + \frac{r}{4}\right)^4 = (\$1000)(1 + 0.025)^4 = \$1103.81$$

Monthly compounding ($n = 12$):

$$A = P \cdot \left(1 + \frac{r}{12}\right)^{12} = (\$1000)(1 + 0.00833)^{12} = \$1104.71$$

Daily compounding ($n = 365$):

$$A = P \cdot \left(1 + \frac{r}{365}\right)^{365} = (\$1000)(1 + 0.000274)^{365} = \$1105.16$$

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n = P \cdot \left(1 + \frac{1}{\frac{n}{r}}\right)^n = P \cdot \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r}\right]^r = P \cdot \left[\left(1 + \frac{1}{h}\right)^h\right]^r$$

$$h = \frac{n}{r}$$

	$\left(1 + \frac{r}{n}\right)^n$			
	$n = 100$	**$n = 1000$**	**$n = 10{,}000$**	**e^r**
$r = 0.05$	1.0512580	1.0512698	1.051271	1.0512711
$r = 0.10$	1.1051157	1.1051654	1.1051704	1.1051709
$r = 0.15$	1.1617037	1.1618212	1.1618329	1.1618342
$r = 1$	2.7048138	2.7169239	2.7181459	2.7182818

Theorem

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = Pe^{rt}$$

EXAMPLE

Using Continuous Compounding

Find the amount A that results from investing a principal P of \$2000 at an annual rate r of 8% compounded continuously for a time t of 1 year.

$$A = Pe^{rt}$$

OBJECTIVE 2

2 **Calculate Effective Rates of Return**

The **effective rate** of interest is the equivalent annual simple rate of interest that would yield the same amount as compounding after 1 year.

	Annual Rate	Effective Rate
Annual compounding	10%	10%
Semiannual compounding	10%	10.25%
Quarterly compounding	10%	10.381%
Monthly compounding	10%	10.471%
Daily compounding	10%	10.516%
Continuous compounding	10%	10.517%

EXAMPLE

Computing the Effective Rate of Interest

On January 2, 2004, \$2000 is placed in an Individual Retirement Account (IRA) that will pay interest of 7% per annum compounded continuously.

(a) What will the IRA be worth on January 1, 2024?

(b) What is the effective rate of interest?

Exploration

For the IRA described in Example 4, how long will it be until $A = \$4000$? \$6000?

[**Hint:** Graph $Y_1 = 2000e^{0.07x}$ and $Y_2 = 4000$. Use INTERSECT to find x.]

OBJECTIVE 3

3 **Determine the Present Value of a Lump Sum of Money**

Theorem

Present Value Formulas

The present value P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

If the interest is compounded continuously, then

$$P = Ae^{-rt}$$

EXAMPLE

Computing the Value of a Zero-Coupon Bond

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for $1000. How much should you be willing to pay for it now if you want a return of

(a) 7% compounded monthly?

(b) 6% compounded continuously?

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

$$P = Ae^{-rt}$$

EXAMPLE

Rate of Interest Required to Double an Investment

What annual rate of interest compounded quarterly should you seek if you want to double your investment in 6 years?

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

OBJECTIVE 4

4 **Determine the Time Required to Double or Triple a Lump Sum of Money**

EXAMPLE

Doubling and Tripling Time for an Investment

(a) How long will it take for an investment to double in value if it earns 6% compounded continuously?

(b) How long will it take to triple at this rate?

$$A = Pe^{rt}$$