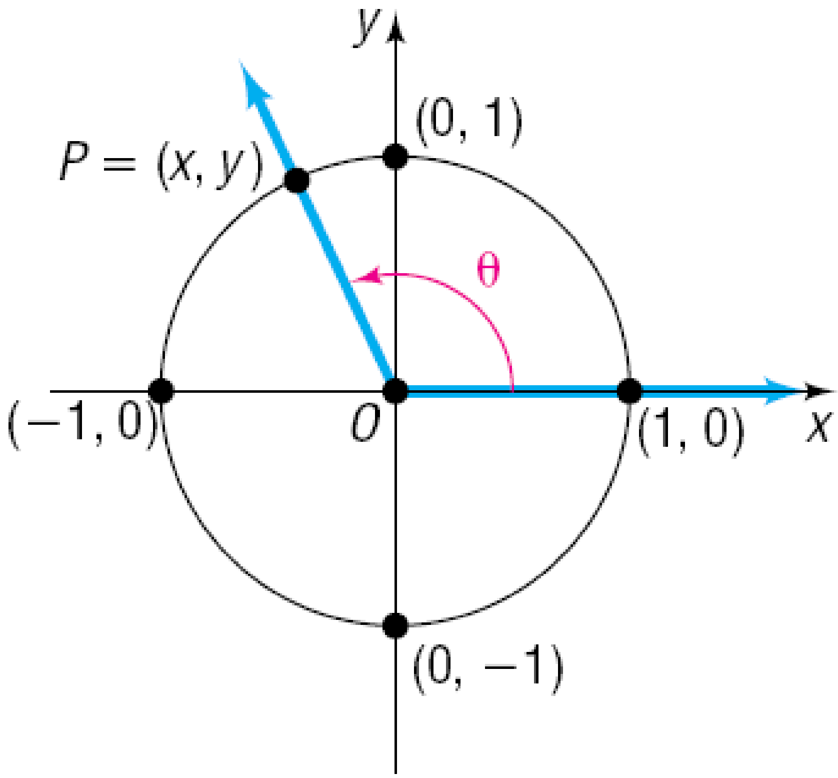
Section 5.3

Properties of the Trigonometric Functions

OBJECTIVE 1

1 **Determine the Domain and the Range of the Trigonometric Functions**



$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x}, \quad x \neq 0$$

$$\csc\theta = \frac{1}{y}, \quad y \neq 0 \qquad \sec\theta = \frac{1}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

The domain of the sine function is the set of all real numbers.

The domain of the cosine function is the set of all real numbers.

The domain of the tangent function is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$ (90°).

The domain of the secant function is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$ (90°).

The domain of the cotangent function is the set of all real numbers, except integer multiples of π (180°).

The domain of the cosecant function is the set of all real numbers, except integer multiples of π (180°).

$$-1 \le \sin\theta \le 1 \quad \text{and} \quad -1 \le \cos\theta \le 1$$

$$\csc\theta \le -1 \quad \text{or} \quad \csc\theta \ge 1$$

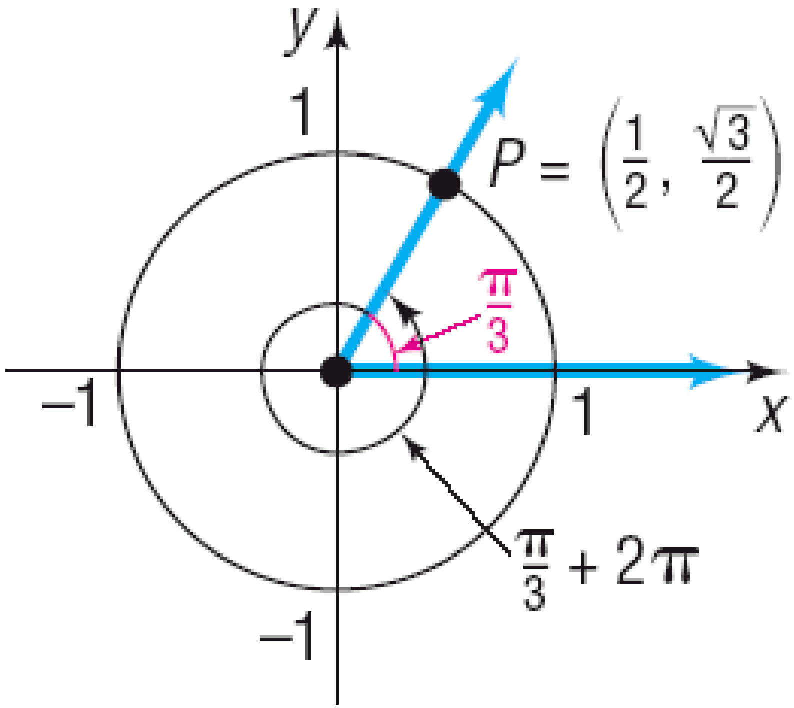
$$\sec\theta \le -1 \quad \text{or} \quad \sec\theta \ge 1$$

$$-\infty < \tan\theta < \infty \quad \text{and} \quad -\infty < \cot\theta < \infty$$

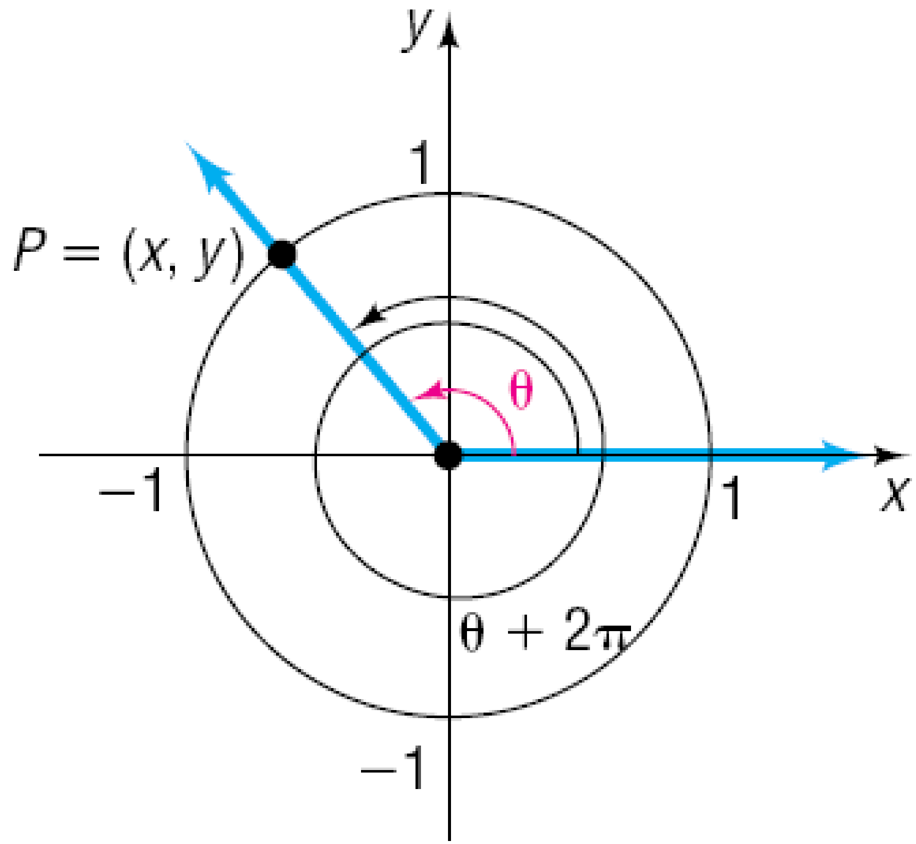
Function	Symbol	Domain	Range
sine	$f(\theta) = \sin \theta$	All real numbers	All real numbers from -1 to 1, inclusive
cosine	$f(\theta) = \cos \theta$	All real numbers	All real numbers from -1 to 1, inclusive
tangent	$f(\theta) = \tan \theta$	All real numbers, except odd multiples of $\frac{\pi}{2}$ (90°)	All real numbers
cosecant	$f(\theta) = \csc \theta$	All real numbers, except integer multiples of π (180°)	All real numbers greater than or equal to 1 or less than or equal to -1
secant	$f(\theta) = \sec \theta$	All real numbers, except odd multiples of $\frac{\pi}{2}$ (90°)	All real numbers greater than or equal to 1 or less than or equal to -1
cotangent	$f(\theta) = \cot \theta$	All real numbers, except integer multiples of π (180°)	All real numbers

OBJECTIVE 2

2 Determine the Period of the Trigonometric Functions



y
1
$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{3}$
–1
1
x
$\frac{\pi}{3} + 2\pi$
–1



y
1
$P = (x, y)$
θ
–1
1
x
θ + 2π
–1

$$\sin(\theta + 2\pi k) = \sin\theta \qquad \cos(\theta + 2\pi k) = \cos\theta$$

where k is any integer

A function f is called **periodic** if there is a positive number p such that, whenever θ is in the domain of f, so is $\theta + p$, and

$$f(\theta + p) = f(\theta)$$

If there is a smallest such number p, this smallest value is called the **(fundamental) period** of f.

Periodic Properties

$$\sin(\theta + 2\pi) = \sin\theta$$
$$\csc(\theta + 2\pi) = \csc\theta$$

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sec(\theta + 2\pi) = \sec\theta$$

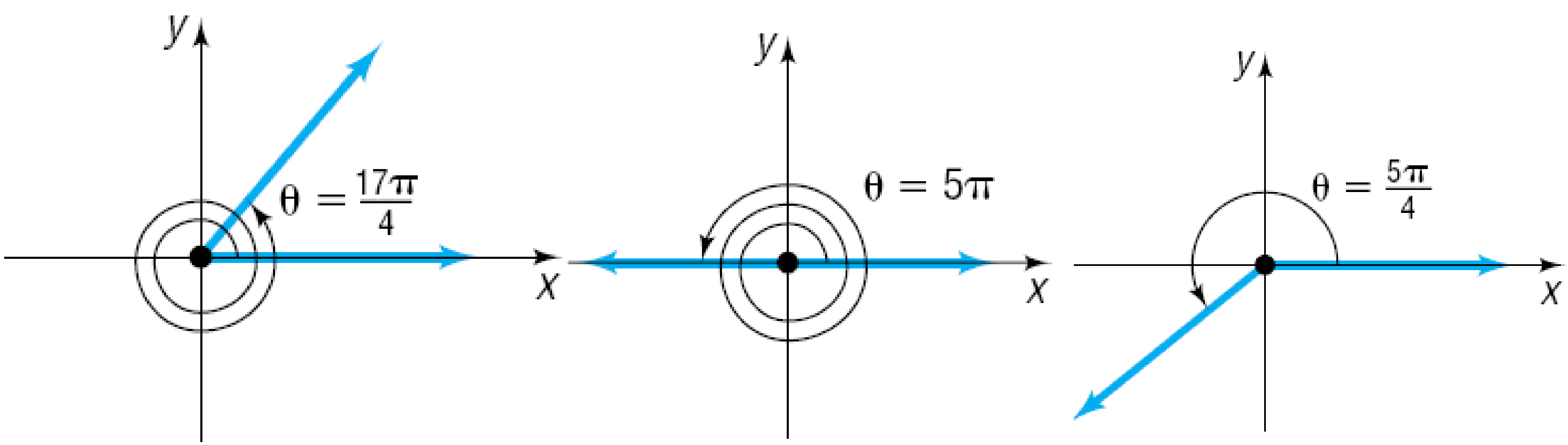
$$\tan(\theta + \pi) = \tan\theta$$
$$\cot(\theta + \pi) = \cot\theta$$

EXAMPLE

Finding Exact Values Using Periodic Properties

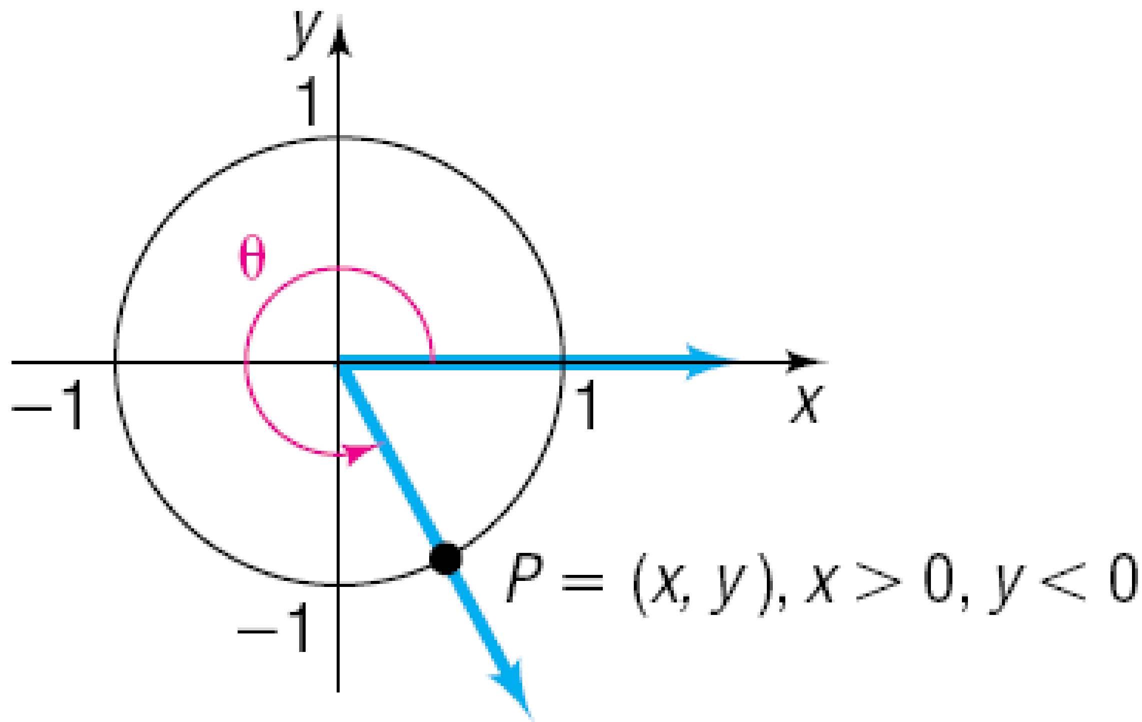
Find the exact value of:

(a) $\sin \frac{17\pi}{4}$ (b) $\cos(5\pi)$ (c) $\tan \frac{5\pi}{4}$



OBJECTIVE 3

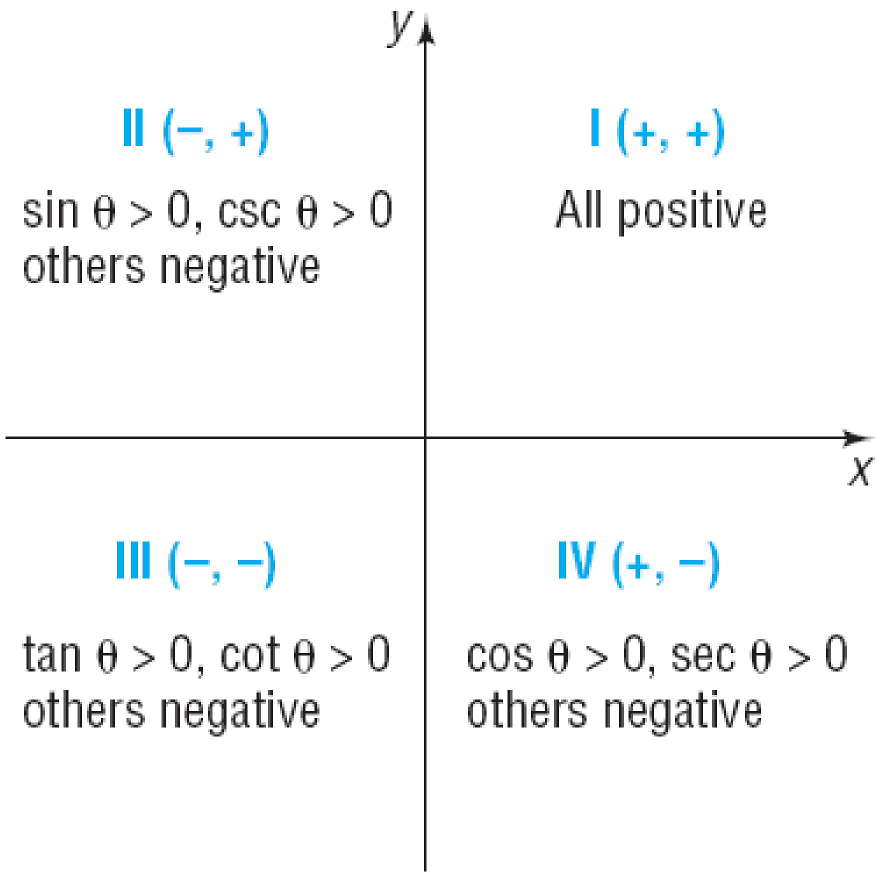
3 Determine the Signs of the Trigonometric Functions in a Given Quadrant

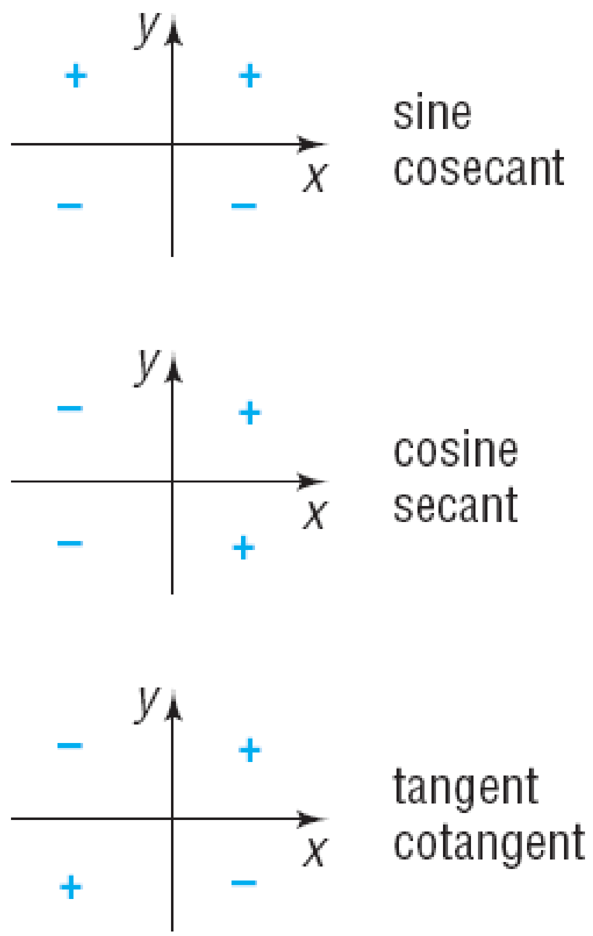


$$\sin\theta = y < 0 \qquad \cos\theta = x > 0 \qquad \tan\theta = \frac{y}{x} < 0$$

$$\csc\theta = \frac{1}{y} < 0 \qquad \sec\theta = \frac{1}{x} > 0 \qquad \cot\theta = \frac{x}{y} < 0$$

Quadrant of θ	$\sin\theta$, $\csc\theta$	$\cos\theta$, $\sec\theta$	$\tan\theta$, $\cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative

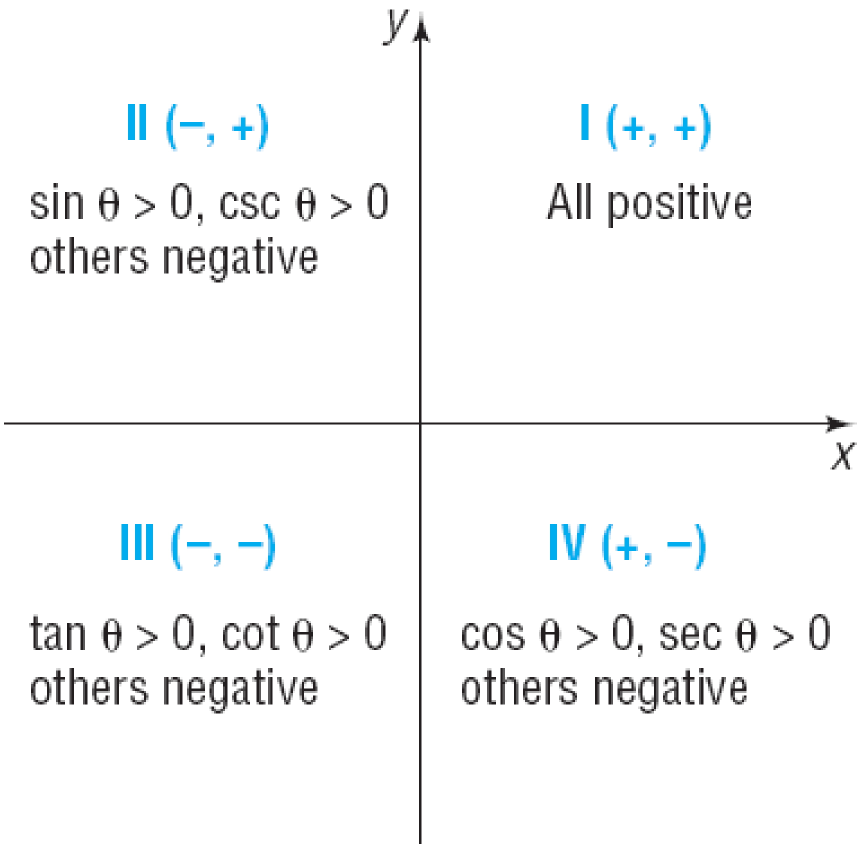




EXAMPLE

Finding the Quadrant in Which an Angle Lies

If $\sin \theta > 0$ and $\cos \theta < 0$, name the quadrant in which the angle θ lies.



OBJECTIVE 4

4 **Find the Values of the Trigonometric Functions Using Fundamental Identities**

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

EXAMPLE

Finding the Values of the Remaining Trigonometric Functions, Given sin θ and cos θ

Given $\sin\theta = \dfrac{\sqrt{10}}{10}$ and $\cos\theta = \dfrac{3\sqrt{10}}{10}$, find the value of each of the four remaining trigonometric functions of θ.

The equation of the unit circle is $x^2 + y^2 = 1$

But $y = \sin\theta$ and $x = \cos\theta$, so

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta}$$

$$\boxed{\tan^2 \theta + 1 = \sec^2 \theta}$$

$$\boxed{\cot^2 \theta + 1 = \csc^2 \theta}$$

Fundamental Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

EXAMPLE

Finding the Exact Value of a Trigonometric Expression Using Identities

Find the exact value of each expression. Do not use a calculator.

$$(a)\ \frac{1}{\csc^2 35^\circ} + \cos^2 35^\circ \qquad (b)\ \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} - \cot\frac{\pi}{3}$$

OBJECTIVE 5

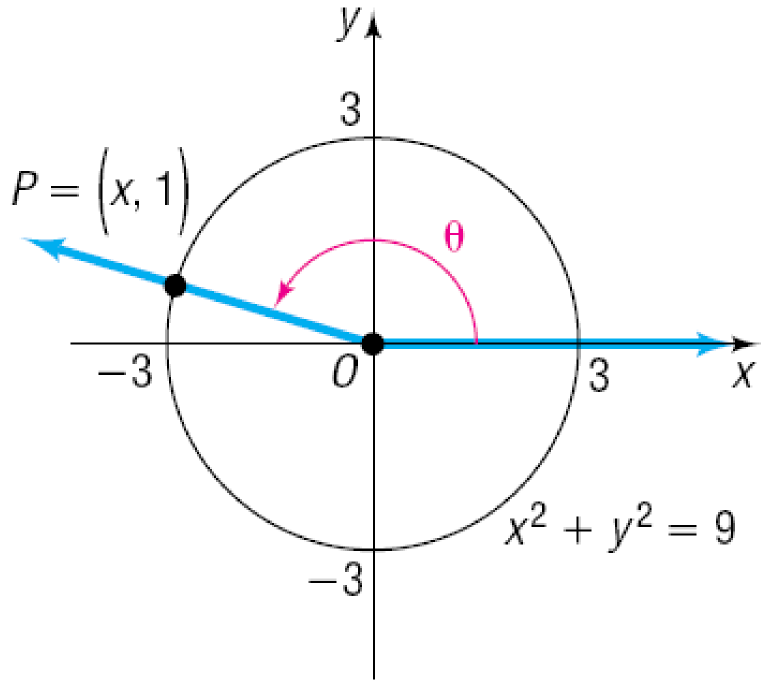
5 **Find the Exact Values of the Trigonometric Functions of an Angle Given One of the Functions and the Quadrant of the Angle**

EXAMPLE

Finding Exact Values Given One Value and the Sign of Another

Given that $\sin\theta = \dfrac{1}{3}$ and $\cos\theta < 0$, find the exact values of each of the remaining five trigonometric functions.

Solution 1 Using a Circle



EXAMPLE

Finding Exact Values Given One Value and the Sign of Another

Given that $\sin\theta = \dfrac{1}{3}$ and $\cos\theta < 0$, find the exact values of each of the remaining five trigonometric functions.

Solution 2 Using Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Finding the Values of the Trigonometric Functions When One Is Known

Given the value of one trigonometric function and the quadrant in which θ lies, the exact value of each of the remaining five trigonometric functions can be found in either of two ways.

Method 1 Using a Circle of Radius r

STEP 1: Draw a circle showing the location of the angle θ and the point $P = (x, y)$ that corresponds to θ. The radius of the circle is $r = \sqrt{x^2 + y^2}$.

STEP 2: Assign a value to two of the three variables x, y, r based on the value of the given trigonometric function and the location of P.

STEP 3: Use the fact that P lies on the circle $x^2 + y^2 = r^2$ to find the value of the missing variable.

STEP 4: Apply the theorem on page 382 to find the values of the remaining trigonometric functions.

Method 2 Using Identities

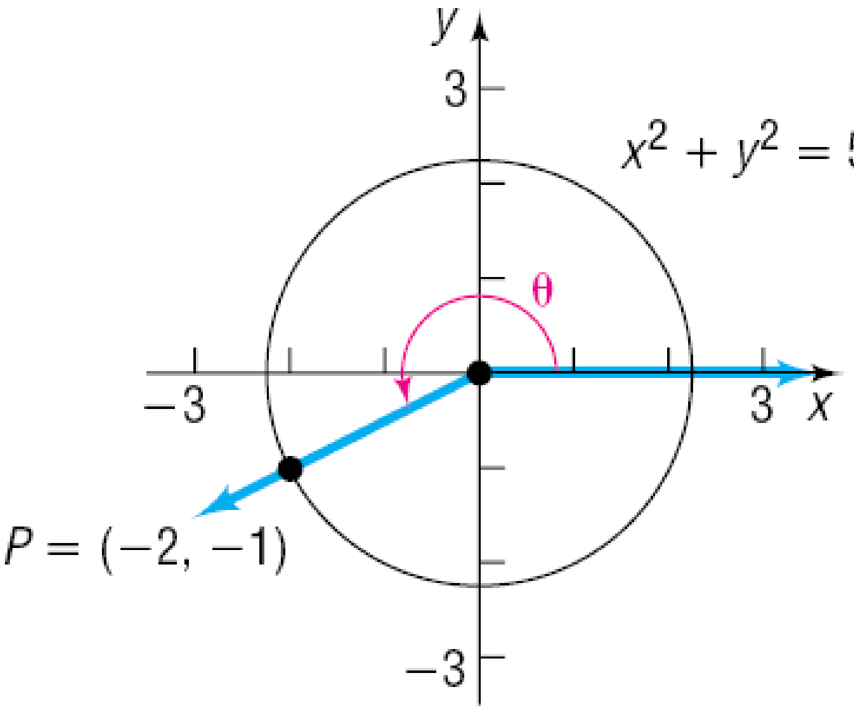
Use appropriately selected identities to find the value of each of the remaining trigonometric functions.

EXAMPLE

Given One Value of a Trigonometric Function, Find the Remaining Ones

Given that $\tan\theta = \dfrac{1}{2}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trigonometric functions of θ.

Solution 1 Using a Circle



EXAMPLE

Given One Value of a Trigonometric Function, Find the Remaining Ones

Given that $\tan\theta = \dfrac{1}{2}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trigonometric functions of θ.

Solution 2
Using Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

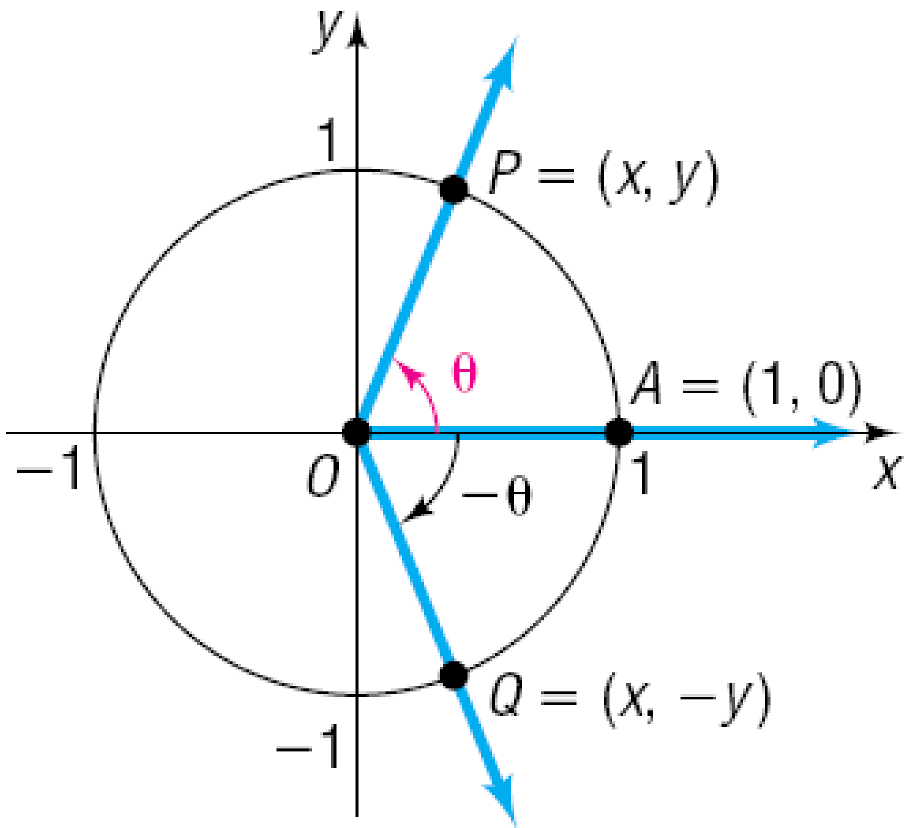
$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

OBJECTIVE 6

6 **Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions**

Even–Odd Properties

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$



EXAMPLE

Finding Exact Values Using Even–Odd Properties

Find the exact value of:

(a) $\cos(-60°)$ (b) $\sin(-390°)$ (c) $\tan\left(-\frac{37\pi}{4}\right)$