4. Find all the points of inflection of $f(x)=x^5e^{-x}$ (3 points)
(Must Justify Your Answer)

I.P are $(0,0)$
$(5-\sqrt{5},\ 10.169)$
$(5+\sqrt{5},\ 14.29)$

$f'(x) = 5x^4e^{-x} + x^5(-e^{-x})$

$f''(x) = 20x^3e^{-x} + -5x^4e^{-x} + 5x^4(-e^{-x}) + x^5(e^{-x})$

$= 20x^3e^{-x} - 10x^4e^{-x} + x^5e^{-x}$

$x^3e^{-x}\left[20 - 10x + x^2\right] = 0$

$x = 0$

$x = \dfrac{-(-10) \pm \sqrt{100 - 4(1)(20)}}{2} = \dfrac{10 \pm \sqrt{20}}{2} = 5 \pm \sqrt{5}$

$f'' = (-)$, $f''(+)$, $f''(-)$, $f''(+)$

$x=0$, $y=0$; $x = 5-\sqrt{5}$, $y = 10.169$; $x = 5+\sqrt{5}$, $y = 14.29$

5. Find the Critical numbers of the function $f(x)=x^{\frac{2}{3}}(x-3)^2$ (3 points)

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x-3)^2 + x^{\frac{2}{3}}(2)(x-3)$

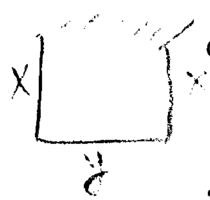
$= \dfrac{2(x-3)^2}{3x^{1/3}} + \dfrac{2(x-3)x^{2/3}}{1} = \dfrac{2(x-3)^2 + 6(x-3)x}{3x^{1/3}}$

$= \dfrac{2(x^2-6x+9) + 6x^2 - 18x}{3x^{1/3}} = \dfrac{8x^2 - 30x + 18}{3x^{1/3}} = \dfrac{2(4x^2 - 15x + 9)}{3x^{1/3}}$

$= \dfrac{2(4x-3)(x-3)}{3x^{1/3}}$

$x = 0$, $x = \frac{3}{4}$, $x = 3$

6. A farmer has 20 feet of fence, and he wishes to make from it a rectangular pen for his pig Wilbur, using a barn as one of the sides. In square feet, what is the maximum area possible for this pen? (5 points)



$20 = 2x + y \Rightarrow y = 20 - 2x$

$A = xy = x(20 - 2x) = 20x - 2x^2$

$A' = 20 - 4x = 0$ $x = 5$ ft

Max Area $= 20(5) - 2(5)^2 = 100 - 50 = 50$ ft^2

Professor Fred Katiraie Calculus I Form B Quiz Six

Name: Solution

1. Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = (x-3)^3$ on the interval $[0,1]$ (3 points)

$f'(x) = 3(x-3)^2$

$f(0) = (-3)^3 = -27$, $f(1) = (-2)^3 = -8$; $\frac{f(b)-f(a)}{b-a} = \frac{-8-(-27)}{1} = 19$

$3(x-3)^2 = 19$

$(x-3)^2 = \frac{19}{3} \Rightarrow x-3 = \pm\sqrt{\frac{19}{3}} \Rightarrow x = 3 \pm \sqrt{\frac{19}{3}}$

$x = 3 - \sqrt{\frac{19}{3}}$

2. Given that the function $f(x) = x^3 + ax^2 + bx$ has critical numbers at $x = 2$, and $x = -1$, find a and b. (3 points)

$f'(x) = 3x^2 + 2ax + b$

$f'(2) = 3(2)^2 + 2a(2) + b \Rightarrow$ $12 + 4a + b = 0$

$f'(-1) = 3(-1)^2 + 2a(-1) + b \Rightarrow$ $3 - 2a + b = 0$

$\Rightarrow$ $12 + 4a + b = 0$, $-3 + 2a - b = 0$

$9 + 6a = 0$

$a = -\frac{9}{6} = -\frac{3}{2}$

$3 - 2\left(-\frac{3}{2}\right) + b = 0 \Rightarrow b = -6$

3. Find the minimum value of the function $f(x) = x \ln x$
(Must Justify Your Answer) (3 points)

$f'(x) = 1 \ln x + x\frac{1}{x} \Rightarrow \ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$, $y = -e^{-1}$

$(e^{-1}, -e^{-1})$

$f''(x) = \frac{1}{x}$ at $x = e^{-1}$ $f''(e^{-1}) = +$

So min of function occurs at $x = e^{-1}$ and it is $-e^{-1}$

2nd derivative $f''(x)$

$0 \qquad 3-\sqrt{3} \qquad 3+\sqrt{3}$

4. Find all the points of inflection of $f(x)=x^3e^{-x}$
(Must Justify Your Answer) (3 points)

$f'(x) = 3x^2e^{-x} - x^3e^{-x}$

$f''(x) = 6xe^{-x} - 3x^2e^{-x} - 3x^2e^{-x} + x^3e^{-x}$

$f''(x) = e^{-x}(x^3 - 6x^2 + 6x)$

$f''(x) = e^{-x}\, x(x^2 - 6x + 6)$

$x = 0 \qquad x = \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$

5. Find the Critical numbers of the function $f(x)=x^{\frac{2}{3}}(x-3)^2$ (3 points)

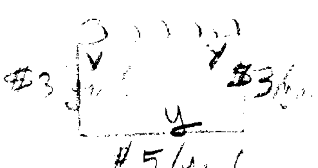
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(x-3)^2 + 2x^{\frac{2}{3}}(x-3)^1$

$= \frac{2(x-3)^2}{3x^{1/3}} + 2x^{\frac{2}{3}}(x-3) = \frac{2(x-3)^2 + 6x(x-3)}{3x^{1/3}}$

$= \frac{2(x^2 - 6x + 9) + 6x^2 - 18x}{3x^{\frac{1}{3}}} = \frac{8x^2 - 30x + 18}{3x^{1/3}} = \frac{2(4x^2 - 15x + 9)}{3x^{\frac{1}{3}}}$

$f'(x) = \frac{2(4x-3)(x-3)}{3x^{1/3}} \Rightarrow$ C.N. are $x = 0,\ x = \frac{3}{4},\ x = 3$

6. Farmer Brown wants to fence in a rectangular plot in a large field, using a rock wall which is already there as the north boundary. The fencing for the east and west sides of the plot will cost \$3/yard, but she needs to use special fencing which will cost \$5/yard on the south side of the plot. If the area of the plot is to be 600 square yards, find the dimensions for the plot which will minimize the cost of the fencing. (5 points)



$xy = 600 \qquad y = \frac{600}{x}$

Cost $= 6x + 5y \Rightarrow$ Cost $= 6x + 5\frac{600}{x} = 6x + 3000x^{-1}$

Cost$' = 6 - 3000x^{-2} = 0 \Rightarrow 6 = \frac{3000}{x^2}$

$6x^2 = 3000 \qquad x^2 = 500$

$x = 10\sqrt{5}$ yards

$y = \frac{600}{x} = \frac{600}{10\sqrt{5}} = \frac{60}{\sqrt{5}} = \frac{60\sqrt{5}}{5} = 12\sqrt{5}$ yards

Professor Fred Katiraie Calculus I; Quiz Six Version A

Name: Solution

$f(x) = (1+x)^{-1}$

$f'(x) = -1(1+x)^{-2}$

1. Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = \dfrac{1}{1+x}$ on the interval $[0,1]$ (3 points)

$f'(c) = \dfrac{-1}{(1+c)^2} = \dfrac{\frac{1}{2}-1}{1}$

$-1 = (1+c)^2\left(-\frac{1}{2}\right)$

$\Rightarrow 2 = (1+c)^2$

$1+c = \pm\sqrt{2}$

$c = -1 \pm \sqrt{2}$

Solution $c = -1+\sqrt{2}$

2. Given that the function $f(x) = x^3 + ax^2 + bx$ has critical numbers at $x = 1$, and $x = -2$, find a and b. (3 points)

$a = \frac{3}{2}$, $b = -6$

$f'(x) = 3x^2 + 2ax + b$

$$\begin{cases} 3(1)^2 + 2a + b = 0 \\ 3(-2)^2 + 2a(-2) + b = 0 \end{cases} \Rightarrow \begin{cases} 3 + 2a + b = 0 \quad (\times -1) \\ 12 - 4a + b = 0 \end{cases} \Rightarrow \begin{cases} -3 - 2a - b = 0 \\ 12 - 4a + b = 0 \end{cases}$$

$9 - 6a = 0$

$a = \frac{9}{6} = \frac{3}{2}$

$3 + 2\left(\frac{3}{2}\right) + b = 0 \Rightarrow b = -6$

3. Find the minimum value of the function $f(x) = x \ln x$
(Must Justify Your Answer) (3 points)

$f'(x) = 1\ln x + x\,\frac{1}{x}$

$f'(x) = \ln x + 1$ let $f'(x) = 0$

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$

$f''(x) = \frac{1}{x}$

at $x = e^{-1}$ $f''(e^{-1}) = \frac{1}{e^{-1}} = e > 0$

The min value is $f(e^{-1}) = e^{-1}\ln e^{-1} = \boxed{-e^{-1}}$