Professor Katiraie Calculus I Summer 2006 Test I (chapters 1 and 2)

Name: Solution Total Possible Points = 150
(Plus 14 pts Extra Credit ☺)

1) Given $f(x)=\dfrac{2}{x-1}$ find $\dfrac{f(x+h)-f(x)}{h}$ (7 Points)

step I) $f(x+h) = \dfrac{2}{x+h-1}$;

step II) $f(x+h)-f(x) = \dfrac{2}{x+h-1} - \dfrac{2}{x-1}$

$= \dfrac{\cancel{2x}-\cancel{2}-\cancel{2x}-2h+\cancel{2}}{(x+h-1)(x-1)} = \dfrac{-2h}{(x+h-1)(x-1)}$

step III) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{-2\cancel{h}}{(x+h-1)(x-1)}\cdot\dfrac{1}{\cancel{h}} = \boxed{\dfrac{-2}{(x+h-1)(x-1)}}$

2) Find the Domain and Range of the following functions: (8 Points)

a) $f(x)=\sqrt{(4-x^2)}$

$4-x^2 \geq 0$

$4 \geq x^2$

Domain $\boxed{-2 \leq x \leq 2}$ $\boxed{\text{Range } [0, 2]}$

b) $g(x)=\ln(\ln(x-6))$

$\ln(x-6) > 0$

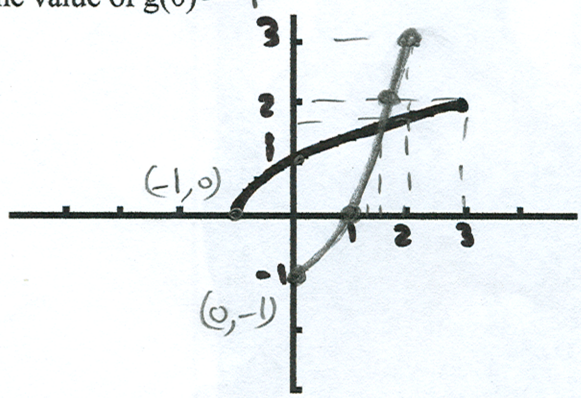
$x-6 > e^0$

$x-6 > 1$

$\boxed{\text{Domain } x > 7}$ $\boxed{\text{Range: } \mathbb{R}}$

3) The graph of g is given. (10 Points)

a) State the value of g(0) = 1



b) Why is g one-to-one? B/c it passes the H. Line Test

c) Estimate the value of $g^{-1}(2)$? = 3

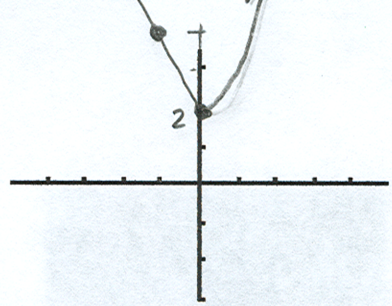
d) Estimate the domain of $g^{-1}(x)$

[0, 2]

e) Sketch the graph of $g^{-1}(x)$

See the graph above

4a) Sketch the graph of the following function: $f(x)=\begin{cases} 2-2x & x<0 \\ e^x+1 & x\geq 0 \end{cases}$ (5 Points)



4b) Discuss **(with reasons)** where the function f(x) is **discontinuous** and why. (5 Points)

The graph is never Discontinuous

because $\lim_{x\to a} f(x) = f(a)$ ie $\lim_{x\to 0} f(x) = f(0) = 2$

5) Determine **(algebraically)** whether f is even, odd, or neither even nor odd (10 Points)

N a) $f(x)=3x^5-4x^3+3$ $f(-x)=3(-x)^5-4(-x)^3+3 = -3x^5+4x^3+3 \neq f(x)$

E b) $f(x)=e^{-x^2}$ $f(-x)=e^{-(-x)^2} = e^{-x^2} = f(x)$

O c) $f(x)=x+\sin(x)$ $f(-x)= -x+\sin(-x) = -x-\sin x = -f(x)$

E d) $f(x)=x^4+2x^2$ $f(-x)=(-x)^4+2(-x)^2 = x^4+2x^2 = f(x)$

6) Solve the following equations algebraically. (10 points)
(Must Show All the Appropriate Steps)

a) $\log x+\log(x+15)=2$

$\log(x(x+15))=2$

$\log(x^2+15x)=2$

$x^2+15x=100$

$x^2+15x-100=0$

$(x+20)(x-5)=0$

$\cancel{x=-20}$ $\boxed{x=5}$

b) $\ln(3+x)-\ln(x-4)=\ln(2)$

$\ln\left(\frac{3+x}{x-4}\right)=\ln 2 \implies \frac{3+x}{x-4}=\frac{2}{1}$

$3+x=2x-8 \implies -x=-11$ $\boxed{x=11}$

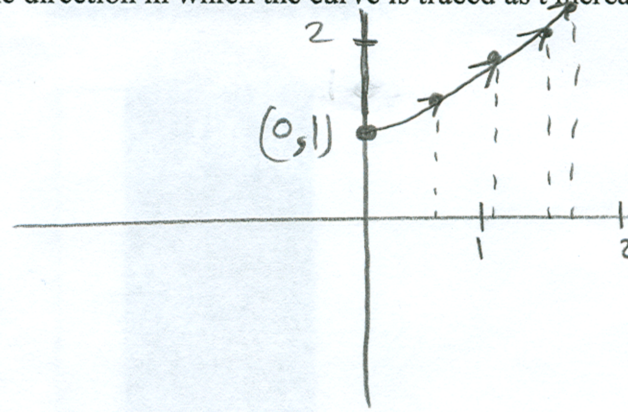
7) If $f(x) = 5x + \ln(x+2)$, find $f^{-1}(-5)$ (5 Points)

$-5 = 5x + \ln(x+2)$

$\boxed{x = -1}$

8a) Sketch the curve represented by the parametric equation $x = \ln(t)$ $y = \sqrt{t}$ $1 \le t \le 5$
And indicate with an arrow the direction in which the curve is traced as t increases. (10 Points)

t	x	y
1	0	1
2	0.693	1.414
3	1.0986	1.7321
4	1.3863	2
5	1.6094	2.2361



8b) Eliminate the parameter to find a Cartesian equation of the curve.

$x = \ln t \Rightarrow t = e^x$

$y = \sqrt{t} = \sqrt{e^x} = \boxed{e^{\frac{x}{2}}}$

8c) State the domain and range of the above graph.

Domain $0 \le x \le 1.6094$ Range $1 \le y \le 2.2361$

9) Let f be a one-to-one function whose inverse function is given by the formula: (10 points)

$$f^{-1}(x) = x^5 + 3x^3 + 2x$$

a) Compute the value of y such that $f^{-1}(y) = 6$ $6 = y^5 + 3y^3 + 2y$ $\boxed{y = 1}$

b) Compute $f^{-1}(-2) = \boxed{-60}$

c) Compute $f(330) = \boxed{3}$ B/c $\boxed{f^{-1}(330) = 3}$

d) Compute the value of x such that $f(x) = 1$

$f^{-1}(1) = 1^5 + 3(1)^3 + 2(1) = 1 + 3 + 2 = \boxed{6}$

10) If an arrow is shot upward on the planet X with a velocity of 60 m/s, its height in meters after t seconds is given by $h(t) = 60t - 2t^2$ (10 Points)

a) Find the average velocity over the given time intervals:

t	y
2	112
2.5	137.5
2.1	117.18
2.01	112.5198
2.001	112.052

i) [2, 2.5] $\text{Avg Velocity} = \frac{137.5-112}{2.5-2} = 51$ m/sec

j) [2, 2.1] $\bar{V} = \frac{117.18-112}{2.1-2} = 51.8$ m/sec

k) [2, 2.01] $\bar{V} = \frac{112.5198-112}{0.01} = 51.98$ m/sec

l) [2, 2.001] $\bar{V} = \frac{112.052-112}{0.001} = 52$ m/sec

b) Find the instantaneous velocity after two seconds. $= 52$ m/sec

11) $f(x) = \begin{cases} x^3+2 & x \le -2 \\ x^2+x+1 & -2 < x < 1 \\ x^4+3 & x \ge 1 \end{cases}$ (10 Points)

Find the following limits (give reasons, if the limit does not exist)

a) $\lim_{x\to -2} f(x) = \boxed{DNE}$

$\lim_{x\to -2^-} f(x) = -8+2 = -6 \neq$

$\lim_{x\to -2^+} f(x) = (-2)^2 + (-2) + 1 = 3$

b) $\lim_{x\to -1} f(x) = (-1)^3 + 2 = \boxed{1}$

c) $\lim_{x\to 1^+} f(x) = 1^4 + 3 = \boxed{4}$

d) $\lim_{x\to 4} f(x) = 4^4 + 3 = \boxed{259}$

12) Find the equation of the exponential function of the form $y = Ca^x$ that passes through the points (0, 4) and (1, 8). (10 Points)

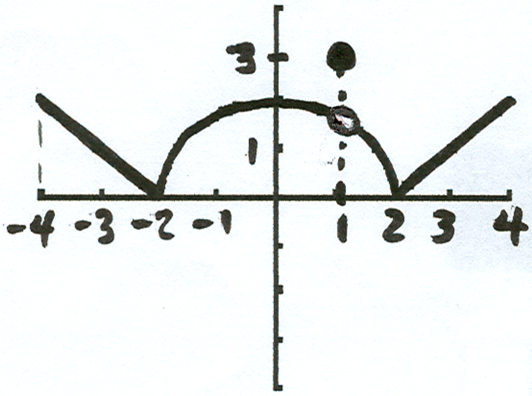
$4 = ca^0 \Rightarrow \boxed{c=4}$

$8 = ca^1$ But $c = 4 \Rightarrow 8 = 4a^1 \Rightarrow \boxed{a=2}$

$\boxed{y = (4)(2)^x}$

13) For the function whose graph is shown below, answer the following equations:

(10 Points)



a) At what number "a" does $\lim_{x\to a} f(x)$ **not** exist? at $a=-4$ and $a=4$

B/C $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

b) At what numbers "a" does $\lim_{x\to a} f(x)$ exists, yet $f(x)$ is **not continuous**? at $a=1$

B/c $\lim_{x\to 1} f(x) = 2 \neq f(1)=3$

c) At what numbers "a" $f(x)$ is continuous, but is **not differentiable**?

at $a=-2$ and $a=+2$

B/c of sharp edges

function is Not Differentiable.

14) Given $f(x)=\begin{cases} 2x^3+16 & x\le -1 \\ x^2+bx+c & -1<x<1 \\ 3x^4-47 & x\ge 1 \end{cases}$ determine the values for b and c so that

$f(x)$ is continuous everywhere.

(10 Points)

$2(-1)^3+16 = -2+16 = 14$

$(-1)^2+b(-1)+c = 14 \longrightarrow \begin{cases} 1-b+c=14 \\ 1+b+c=-44 \end{cases}$

$1+b+c=-44$

$3(1)^4-47=-44$

ADD

$2+2c=-30 \Rightarrow \boxed{c=-16}$

$1-b+c=14$

$1-b+(-16)=14 \Rightarrow \boxed{b=-29}$

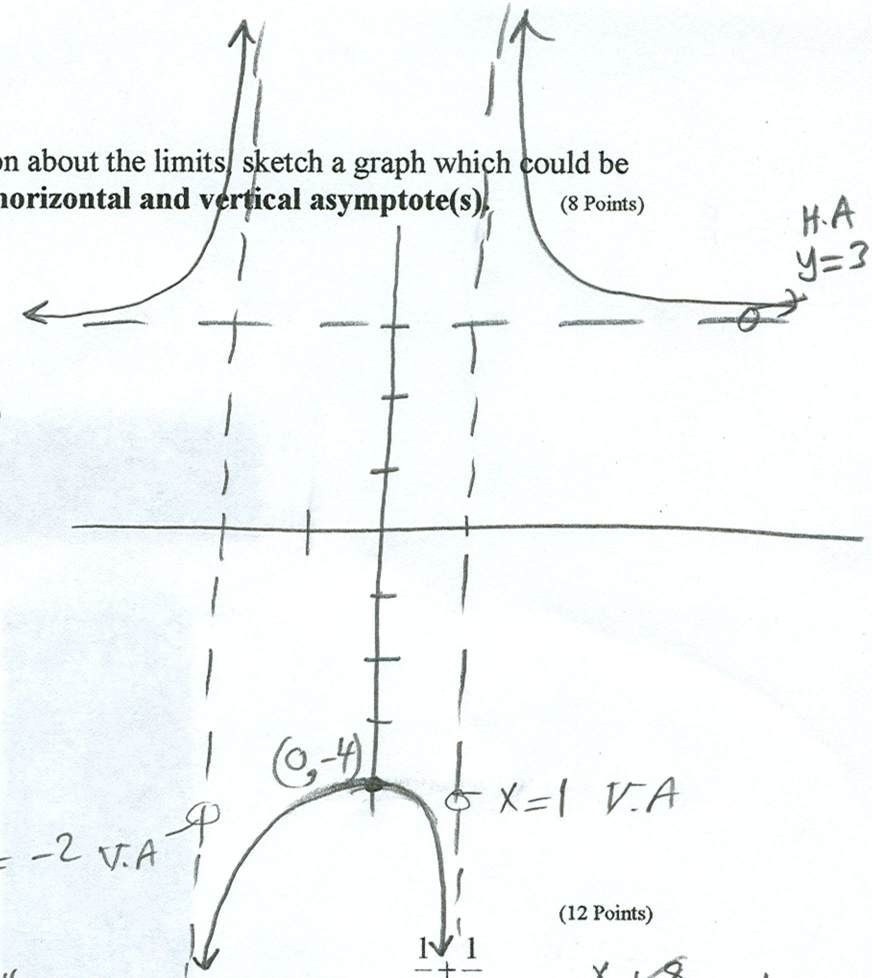
15) Given the following information about the limits, sketch a graph which could be the graph of y = $f(x)$. **Label all horizontal and vertical asymptote(s).** (8 Points)

$\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = 3$

$\lim_{x\to -2^+} f(x) = \lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to -2^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$

$f(0) = -4$



16) Find the following limits: (12 Points)

a) $\lim_{t\to 13} \dfrac{\sqrt{t+3}-4}{t-13} \cdot \dfrac{\sqrt{t+3}+4}{\sqrt{t+3}+4}$

$= \dfrac{t+3-16}{(t-13)(\sqrt{t+3}+4)} = \lim_{t\to 13} \dfrac{1}{\sqrt{t+3}+4} = \boxed{\dfrac{1}{8}}$

b) $\lim_{x\to -8} \dfrac{\frac{1}{8}+\frac{1}{x}}{8+x} = \dfrac{\cancel{x+8}}{8x} \cdot \dfrac{1}{\cancel{x+8}}$

$= \lim_{x\to -8} \dfrac{1}{8x} = \boxed{\dfrac{-1}{64}}$

c) $\lim_{x\to\infty} (\sqrt{x^2+2x}-x)$

$\dfrac{(\sqrt{x^2+2x}-x)(\sqrt{x^2+2x}+x)}{(\sqrt{x^2+2x}+x)}$

$= \dfrac{\cancel{x^2}+2x-\cancel{x^2}}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \dfrac{+2x}{\sqrt{x^2+2x}+x}$

$= \dfrac{\frac{2x}{x}}{\sqrt{\frac{x^2+2x}{x^2}}+\frac{x}{x}} = \lim_{x\to\infty} \dfrac{2}{\sqrt{1+\frac{2}{x}}+1} = \boxed{\dfrac{2}{2} = 1}$

d) $\lim_{x\to\infty} \dfrac{-x-2x^2+6}{3+4x+13x^2} \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$

$= \lim_{x\to\infty} \dfrac{\frac{-x}{x^2}-\frac{2x^2}{x^2}+\frac{6}{x^2}}{\frac{3}{x^2}+\frac{4x}{x^2}+\frac{13x^2}{x^2}} = \boxed{\dfrac{-2}{13}}$

(Extra Credit 3 Points)

17) Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points (2 , 3) and (4 , -5). Find the following:

a) find $f'(3) = \frac{-5-3}{4-2} = \frac{-8}{2} = -4$

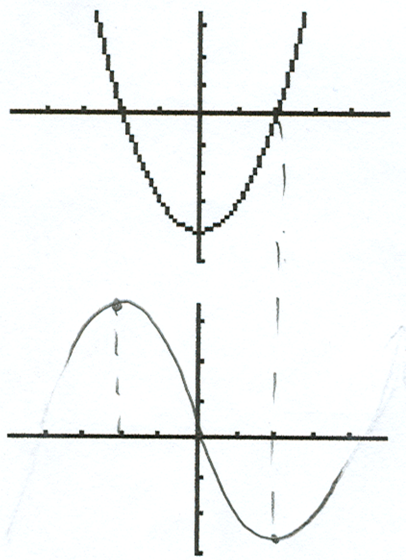
b) find $f(3) = -4(3)+11 = \boxed{-1}$

c) Find an equation of the line tangent to f at $x = 3$

$y-3 = -4(x-2) \Rightarrow y = -4x+11$

(Extra Credit 3 Points)

18) Given the graph of $y = f'(x)$, sketch the graph of $y = f(x)$



(Extra Credit 3 Points)

19) Find the following limit $\lim_{x\to\infty} \frac{\cos x}{x^4}$

(Hint: Use the Squeeze Theorem)

$-1 \leq \cos x \leq 1$

$\frac{-1}{x^4} \leq \frac{\cos x}{x^4} \leq \frac{1}{x^4}$

$\lim_{x\to\infty} \frac{-1}{x^4} = 0$ and $\lim_{x\to\infty} \frac{1}{x^4} = 0$

$\to$ Then by Squeeze theorem $\lim_{x\to\infty} \frac{\cos x}{x^4} = 0$

20) (Extra Credit 5 Points)

Given $f(x) = \sqrt{x-3}$

Find the $f'(x)$ using either of the two definitions discussed in class.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \cdot \frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \lim_{h \to 0} \frac{(x+h-3) - (x-3)}{h(\sqrt{x+h-3} + \sqrt{x-3})} =$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} \cancel{-3} \cancel{-x} \cancel{+3}}{\cancel{h}(\sqrt{x+h-3} + \sqrt{x-3})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \frac{1}{\sqrt{x-3} + \sqrt{x-3}} = \boxed{\frac{1}{2\sqrt{x-3}}}$$