Professor Katiraie Calculus I Summer 2006 Form B Test I (chapters 1 and 2)

Name: Solution Total Possible Points = 150

(Plus 14 pts Extra Credit ☺)

1) Given $f(x) = \frac{7}{x+1}$ find $\frac{f(x+h)-f(x)}{h}$ (7 Points)

$f(x+h) = \frac{7}{x+h+1}$

$f(x+h) - f(x) = \frac{7}{x+h+1} - \frac{7}{x+1} = \frac{7x+7-(7x+7h+7)}{(x+h+1)(x+1)}$

$= \frac{\cancel{7x}+\cancel{7}-\cancel{7x}-7h-\cancel{7}}{(x+h+1)(x+1)} = \frac{-7h}{(x+h+1)(x+1)}$

$\frac{f(x+h)-f(x)}{h} = \frac{-7h}{(x+h+1)(x+1)} \cdot \frac{1}{h} = \boxed{\frac{-7}{(x+h+1)(x+1)}}$

2) Find the Domain and Range of the following functions: (8 Points)

a) $f(x) = \sqrt{(16-x^2)}$

Domain $-4 \le x \le 4$

Range $0 \le y \le 4$

b) $g(x) = \ln(\ln(x+6))$

Domain $\ln(x+6) > 0$

$x+6 > e^0$

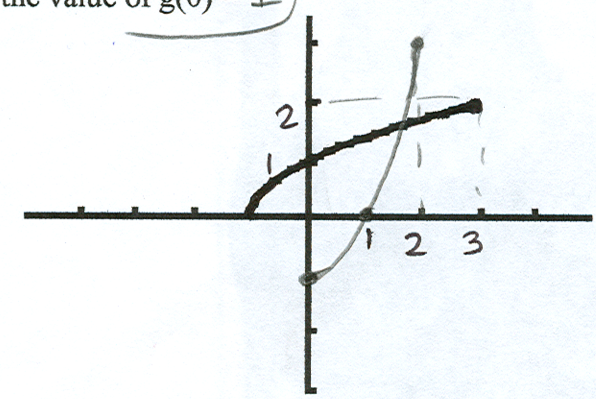
$x+6 > 1$

$\boxed{x > -5}$

$\boxed{\text{Range } \mathbb{R}}$

3) The graph of g is given. (10 Points)

a) State the value of $\boxed{g(0) = 1}$



b) Why is g one-to-one? Because the graph passes Horizontal line test

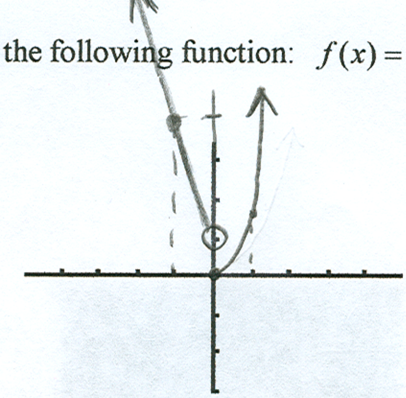
c) Estimate the value of $g^{-1}(2)$? $= 3$

d) Estimate the domain of $g^{-1}(x)$

$[0, 2]$

e) Sketch the graph of $g^{-1}(x)$

4a) Sketch the graph of the following function: $f(x)=\begin{cases}1-2x & x<0\\ e^x-1 & x\ge 0\end{cases}$ (5 Points)



4b) Discuss **(with reasons)** where the function f(x) is **discontinuous** and why. (5 Points)

f(x) is discontinuous at $x=0$ Because $\lim_{x\to 0} f(x)$ DNE

$\lim_{x\to 0} f(x) \neq f(0)$ $f(0)=0$

5) Determine **(algebraically)** whether f is even, odd, or neither even nor odd (10 Points)

N a) $f(x)=3x^5-4x^2+3$ $f(-x)=3(-x)^5-4(-x)^2+3\neq f(x)$
$\neq -f(x)$

N b) $f(x)=e^{-x}$ $f(-x)=e^{-(-x)}=e^{x}\neq f(x)$
$\neq -f(x)$

O c) $f(x)=x^3+\sin(x)$ $f(-x)=(-x)^3+\sin(-x)=-x^3-\sin(x)=-f(x)$

E d) $f(x)=x^4+2x^2$ $f(-x)=(-x)^4+2(-x)^2=x^4+2x^2=f(x)$

6) Solve the following equations algebraically. (10 points)
(Must Show All the Appropriate Steps)

a) $\log x+\log(x+3)=1$ $\log(x^2+3x)=1$

$x^2+3x=10 \Rightarrow x^2+3x-10=0$
$(x+5)(x-2)=0$

~~$x=-5$~~ $x=2$

b) $\ln(3-x)-\ln(x+4)=\ln(2)$

$\frac{3-x}{x+4}=2$

$2x+8=3-x \Rightarrow 3x=-5$ $\boxed{x=\frac{-5}{3}}$

2

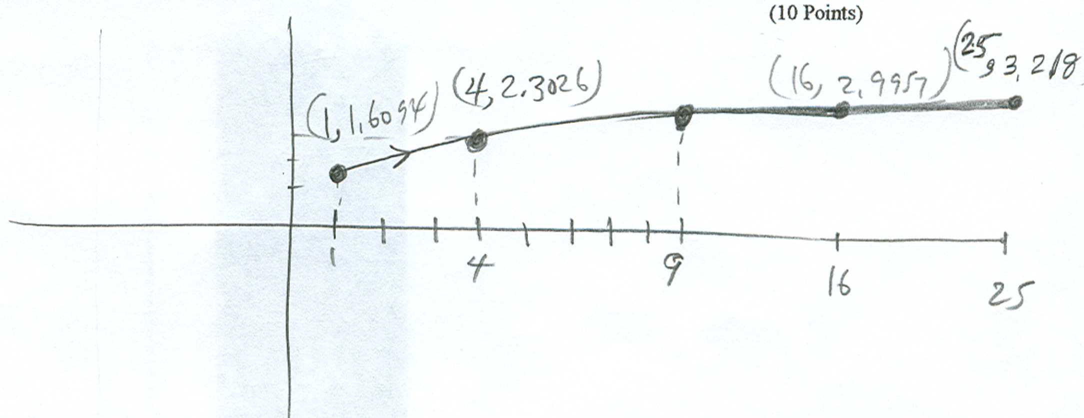
7) If $f(x)=5x+\ln(x+2)$, find $f^{-1}(-2)$ (5 Points)

$-2 = 5x + \ln(x+2)$

$x = -0.4833$

8a) Sketch the curve represented by the parametric equation
$x=t^2 \quad y= \ln(5t) \quad 1\le t\le 5$
And indicate with an arrow the direction in which the curve is traced as t increases.
(10 Points)

t	x	y
1	1	1.6094
2	4	2.3026
3	9	2.7081
4	16	2.9957
5	25	3.2189



8b) Eliminate the parameter to find a Cartesian equation of the curve.

$t=\sqrt{x} \Longrightarrow \boxed{y=\ln(5\sqrt{x})}$

8c) State the domain and range of the above graph.

Domain $1 \le x \le 25$

Range $1.6094 \le y \le 3.2189$

9) Let f be a one-to-one function whose inverse function is given by the formula: (10 points)
$$f^{-1}(x)=x^5+3x^3+2x$$

a) Compute the value of y such that $f^{-1}(y)=6$ $\boxed{y=1}$

b) Compute $f^{-1}(-2)$ $= \boxed{-60}$

c) Compute $f(326)$ $= \boxed{2.9918}$

d) Compute the value of x such that $f(x)=1$

$f^{-1}(1)=6 \Longrightarrow \boxed{x=6}$

t	y
2	92
2.5	112.5
2.1	96.18
2.01	92.4198
2.001	92.041998

10) If an arrow is shot upward on the planet X with a velocity of 50 m/s, its height in meters after t seconds is given by $h(t) = 50t - 2t^2$ (10 Points)

a) Find the average velocity over the given time intervals:

i) $[2, 2.5]$ $\Rightarrow$ $V_{avg} = \bar{V} = \frac{112.5-92}{2.5-2} = \boxed{41 \text{ m/sec}}$

j) $[2, 2.1]$ $\Rightarrow$ $\bar{V} = \frac{96.18-92}{2.1-2} = \boxed{41.8 \text{ m/sec}}$

k) $[2, 2.01]$ $\Rightarrow$ $\bar{V} = \frac{92.4198-92}{2.01-2} = \boxed{41.98 \text{ m/sec}}$

l) $[2, 2.001]$ $\Rightarrow$ $\bar{V} = \frac{92.041998-92}{0.001} = \boxed{41.998 \text{ m/sec}}$

b) Find the instantaneous velocity after two seconds. $\boxed{= 42 \text{ m/sec}}$

11) $f(x) = \begin{cases} x^3+2 & x \le -2 \\ x^2+x+1 & -2 < x < 1 \\ x^4+3 & x \ge 1 \end{cases}$ (10 Points)

Find the following limits (give reasons, if the limit does not exist)

a) $\lim_{x\to -2} f(x) = \begin{cases} (-2)^3+2 = -6 \\ (-2)^2+(-2)+1 = 3 \end{cases}$ $\Rightarrow$ DNE

b) $\lim_{x\to -1} f(x) = (-1)^3+2 = \boxed{1}$

c) $\lim_{x\to 1^+} f(x) = 4$ $\quad 1+3=4$

d) $\lim_{x\to 4} f(x) = 4^4+3$ $\boxed{=259}$

12) Find the equation of the exponential function of the form $y = Ca^x$ that passes through the points (0, 5) and (1, 15). (10 Points)

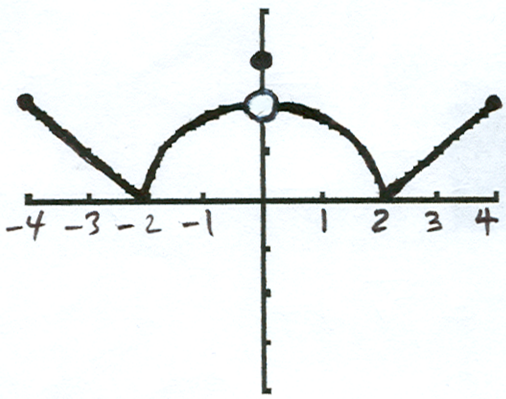
$(0, 5) \Rightarrow 5 = Ca^0 \Rightarrow C = 5$

$(1, 15) \Rightarrow 15 = 5a^1 \Rightarrow \boxed{a = 3}$

$\boxed{y = 5(3)^x}$

13) For the function whose graph is shown below, answer the following equations:

(10 Points)



a) At what number "a" does $\lim_{x \to a} f(x)$ **not** exist? $a = -4$ and $a = 4$

b) At what numbers "a" does $\lim_{x \to a} f(x)$ exists, yet $f(x)$ is **not continuous**? $a = 0$

c) At what numbers "a" $f(x)$ is continuous, but is **not differentiable**? $a = -2$ and

$a = 2$

14) Given $f(x) = \begin{cases} 2x^3 + 8 & x \le -1 \\ x^2 + bx + c & -1 < x < 1 \\ 3x^4 - 9 & x \ge 1 \end{cases}$ determine the values for b and c so that $f(x)$ is continuous everywhere.

(10 Points)

$2(-1)^3 + 8 = 6$

$(-1)^2 + b(-1) + c = 6 \Rightarrow 1 - b + c = 6$ $\begin{cases} -b + c = 5 \\ b + c = -7 \end{cases}$

$1 + b + c = -6$ $1 + b + c = -6$

$3(1)^4 - 9 = -6$ $2c = -2$

$\boxed{b = -6} \longleftarrow \boxed{c = -1}$

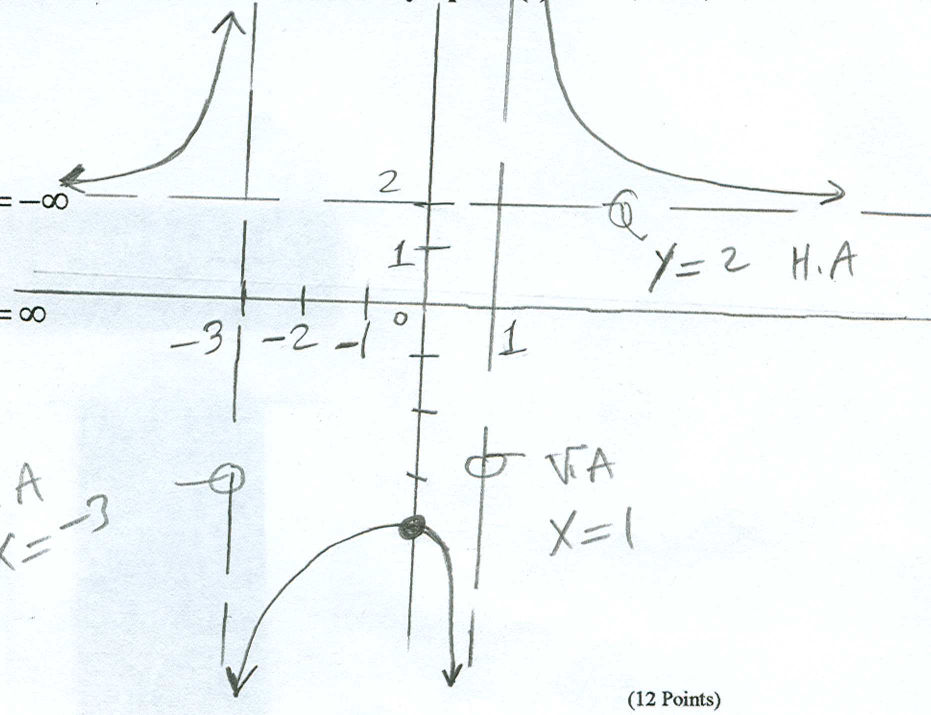
15) Given the following information about the limits, sketch a graph which could be the graph of y = $f(x)$. **Label all horizontal and vertical asymptote(s).** (8 Points)

$\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = 2$

$\lim_{x\to -3^+} f(x) = \lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to -3^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$

$f(0) = -4$



16) Find the following limits: (12 Points)

a) $\lim_{t\to 3} \frac{\sqrt{t+6}-3}{t-3} \cdot \frac{\sqrt{t+6}+3}{\sqrt{t+6}+3}$

$= \frac{t+6-9}{(t-3)\sqrt{t+6}+3} = \lim_{t\to 3} \frac{1}{\sqrt{3+6}+3} = \boxed{\frac{1}{6}}$

b) $\lim_{x\to -2} \frac{\frac{1}{2}+\frac{1}{x}}{2+x} = \frac{\frac{x+2}{2x}}{x+2}$

$\lim_{x\to -2} \frac{\cancel{x+2}}{2x} \cdot \frac{1}{\cancel{x+2}}$

$= \frac{-1}{4}$

c) $\lim_{x\to\infty} (\sqrt{x^2-3x}-x) \cdot \frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x}$

$= \lim_{x\to\infty} \frac{x^2-3x-x^2}{\sqrt{x^2-3x}+x} = \frac{-3x}{\sqrt{x^2-3x}+x}$

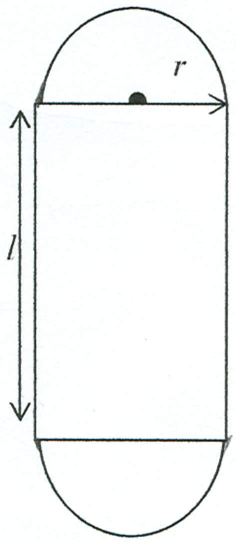
$= \lim_{x\to\infty} \frac{\frac{-3x}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{3x}{x^2}}+\frac{x}{x}} = \lim_{x\to\infty} \frac{-3}{\sqrt{1-\frac{3}{x}}+1} = \boxed{\frac{-3}{2}}$

d) $\lim_{x\to\infty} \frac{-x-2x^2+6}{3+4x+14x^2} \quad \frac{\div x^2}{\div x^2}$

$\lim_{x\to\infty} \frac{\frac{-x}{x^2}-\frac{2x^2}{x^2}+\frac{6}{x^2}}{\frac{3}{x^2}+\frac{4x}{x^2}+\frac{14x^2}{x^2}} = \boxed{\frac{-2}{14} = \frac{-1}{7}}$

(Extra Credit 3 Points)

17) A field has the shape of a rectangle with a semicircle at each end. The length of the rectangular portion of the field is l, and the radius of each semicircle is r. If the outside perimeter of the field is 250 meters, express the **area** of the field as a function of r, and simplify your answer.



$$2L + 2\pi r = 250 \Rightarrow L + \pi r = 125$$

$$L = 125 - \pi r$$

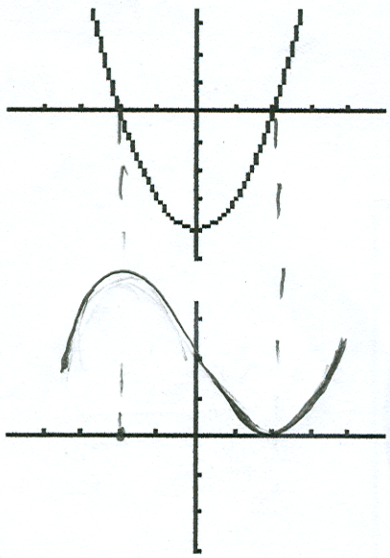
$$A = 2r * L + \pi r^2$$

$$= 2r(125 - \pi r) + \pi r^2$$

$$= 250r - 2\pi r^2 + \pi r^2 = \boxed{250r - \pi r^2 \ (m^2)}$$

(Extra Credit 3 Points)

18) Given the graph of $y = f'(x)$, sketch the graph of $y = f(x)$



(Extra Credit 3 Points)

19) Find the following limit $\lim_{x\to\infty} \frac{\cos 4x}{x^{84}}$

(Hint: Use the Squeeze Theorem)

$$-1 \le \cos 4x \le 1$$

$$\frac{-1}{x^{84}} \le \frac{\cos 4x}{x^{84}} \le \frac{1}{x^{84}}$$

$$\lim_{x\to\infty} \frac{-1}{x^{84}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x^{84}} = 0$$

then by Squeeze theorem

$$\boxed{\lim_{x\to\infty} \frac{\cos 4x}{x^{84}} = 0}$$

20) (Extra Credit 5 Points)

Given $f(x) = \sqrt{2x}$

Find the $f'(x)$ using either of the two definitions discussed in class.

$$\lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \cdot \frac{\sqrt{2(x+h)} + \sqrt{2x}}{\sqrt{2(x+h)} + \sqrt{2x}}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} - \cancel{2x}}{\cancel{h}\left(\sqrt{2(x+h)} + \sqrt{2x}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2(x+h)} + \sqrt{2x}} =$$

$$= \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{2}{2\sqrt{2x}}$$

$$= \frac{1}{\sqrt{2x}}$$