Professor Katiraie Calculus I Summer 2007 Test II (chapter 1 - 3)

Name: Solution Total Possible Points = 150 Plus 10 pts Extra Credits

This is not a graphing calculator test. I will not give credit to answers not supported by your work, i.e. No Procedure = No Points

(10 Points) 1) Solve the following equations algebraically. (10 points)
(Must Show All the Appropriate Steps)

a) $\log x + \log(x+3) = 1$

$\ln(x(x+3)) = 1$

$x^2 + 3x = 10 \Rightarrow x^2 + 3x - 10 = 0$

$(x+5)(x-2) = 0$

$\cancel{x = -5}$ $\boxed{x = 2}$

b) $\ln(3-x) - \ln(x+4) = \ln(2)$

$\ln\left(\frac{3-x}{x+4}\right) = \ln 2$ $\frac{3-x}{x+4} = 2 \Rightarrow 2x + 8 = 3 - x$

$3x = -5 \Rightarrow \boxed{x = \frac{-5}{3}}$

2) Let f be a one-to-one function whose inverse function is given by the formula:

$f^{-1}(x) = x^5 + 2x^3 + 3x + 1$ (10 Points)

a) Compute $f^{-1}(-5) = \boxed{-3389}$

b) Compute $f(5)$ $5 = x^5 + 2x^3 + 3x + 1 \Rightarrow \boxed{x = 0.83}$

c) Compute the value of x such that $f(x) = 4$ $f^{-1}(4) = \boxed{1165}$

d) Compute the value of y such that $f^{-1}(y) = 7 \Rightarrow \boxed{y = 1}$

3) If an arrow is shot upward on the moon with a velocity of 60 m/s, its height in meters after t seconds is given by $h(t) = 60t - 0.83t^2$ (10 Points)

t	$h(t)$
1	59.17
1.5	88.13
1.001	59.23

A) Find the average velocity over the given time intervals:

i) [1,1.5] $\frac{88.1325 - 59.17}{0.5} = \boxed{57.925 \text{ m/sec}}$

j) [1,1.001] $\frac{59.23 - 59.17}{0.001} = \boxed{60 \text{ m/sec}}$

B) Find the instantaneous velocity after one second.

$h'(t) = 60 - 2(0.83)t \Rightarrow \boxed{h'(1) = 58.34 \text{ m/sec}}$

4) Find the following limits

(Must Show Procedure, No Procedure = No Points) (10 Points)

a) $\lim_{x\to\infty} \frac{1-2x^2}{x^2+x}$ $\frac{\div x^2}{\div x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{2x^2}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2}}$

$\boxed{= -2}$

b) $\lim_{x\to\infty}(\sqrt{x^2+2x} - x) \cdot \frac{(\sqrt{x^2+2x} + x)}{(\sqrt{x^2+2x} + x)}$

$= \lim_{x\to\infty} \frac{\cancel{x^2} + 2x - \cancel{x^2}}{\sqrt{x^2+2x} + x} = \boxed{1}$

c) $\lim_{t\to 0} x^4 \cos\left(\frac{1}{x^2}\right)$

$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

$-x^4 \le x^4 \cos\left(\frac{1}{x^2}\right) \le +x^4$

$\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0 \Rightarrow$ By Sq. Theorem $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x^2}\right) = 0$

d) $\lim_{t\to 0} \frac{1}{t} - \frac{1}{t^2+t} = \frac{\cancel{t}+1-1}{t(t+1)}$

$= \lim_{t\to 0} \frac{1}{t+1} = \boxed{1}$

5) Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points (- 2 , 5) and (4 , - 4). Find the following:

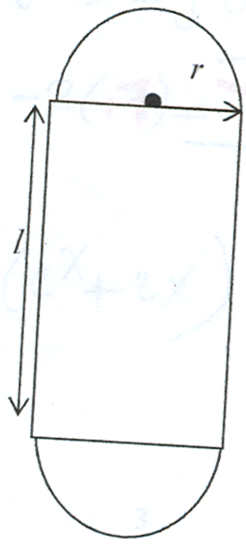
(6 Points)

a) find $f'(3) = \frac{-4-5}{4+2} = \frac{-9}{6} = \frac{-3}{2}$ b) find $f(3) = -\frac{3}{2}(5) + 5 = \boxed{-2.5}$

b) Find an equation of the line tangent to f at $x = 3$

$y - 5 = -\frac{3}{2}(x+2) \Rightarrow y = -\frac{3}{2}(x+2) + 5 = \boxed{-\frac{3}{2}x + 2}$

6) A field has the shape of a rectangle with a semicircle at each end. The length of the rectangular portion of the field is l, and the radius of each semicircle is r. If the outside perimeter of the field is 250 meters, express the **area** of the field as a function of r, and simplify your answer. (4 Points)



$2L + 2\pi r = 250 \Rightarrow L + \pi r = 125$

$L = 125 - \pi r$

$A = \pi r^2 + L \cdot (2r)$

$= \pi r^2 + 2r(125 - \pi r)$

$= \pi r^2 + 250r - 2\pi r^2 = \boxed{-\pi r^2 + 250r \ (m^2)}$

6) Find the derivative of the following functions: **(4 Points each)**
(Do Not Simplify)

a. $f(x)=\dfrac{x^3+4x^2+3}{\sqrt{x}}$ $= x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

$f'(x) = \frac{5}{2}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

b. $u=\sqrt[3]{t^2}+2\sqrt{t^3}$ $= t^{\frac{2}{3}} + 2t^{\frac{3}{2}}$

$u' = \frac{2}{3}t^{-\frac{1}{3}} + 3t^{\frac{1}{2}}$

c. $y(z)=ae^z+\dfrac{b}{z}+\dfrac{c}{z^2}$ $= ae^z + bz^{-1} + cz^{-2}$ (assume a, b, and c are constants)

$y' = ae^z - bz^{-2} - 2cz^{-3}$

d. $y=\sin\left((5x)^3\right)$

$y' = \cos\left((5x)^3\right) 3(5x)^2 (5)$

e. If $f(x)=-2e^x g(x)$ $f'(x) = -2e^x g(x) - 2e^x g'(x)$

$g(0)=2$ and $g'(0)=5$, find $f'(0)$. $= -2e^x(g(x) - g'(x))$

$f'(0) = -2e^0(g(0) + g'(0)) = -2e^0(2+5) = -2(7) = \boxed{-14}$

f. $f(x)=\sin^{-1}\left(e^x+x^2\right)$ $f'(x) = \dfrac{1}{\sqrt{1-(e^x+x^2)^2}} \cdot (e^x + 2x)$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{0(\cos x) + \sin x\,(1)}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \sec x$$

Find the derivative of the following functions: **(4 Points each)**
(Do Not Simplify)

g. $y = \ln(1 + \sec x)$ $\Rightarrow$ $y' = \dfrac{1}{1+\sec x} \cdot (\sec x \tan x)$

h. $f(\theta) = \sqrt[3]{1 + \tan\theta}$ $\Rightarrow y' = \frac{1}{3}(1+\tan\theta)^{-\frac{2}{3}}(\sec^2\theta)$

i. $y = \sin(\tan\sqrt{1+x^2})$ $\quad y' = \cos\left(\tan(1+x^2)^{\frac{1}{2}}\right)\sec^2\left((1+x^2)^{\frac{1}{2}}\right)\frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$

(10 Points) 7) Find the equation of the tangent line to the curve $\sqrt{x} + \sqrt{y} = 9$ at the point (16 , 25).

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 9$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\, y' = 0$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\, y' = 0 \Longrightarrow \frac{1}{2\sqrt{16}} + \frac{1}{2\sqrt{25}}\, y' = 0 \qquad y' = \frac{-5}{4}$$

$$y - 25 = \frac{-5}{4}(x - 16) \Rightarrow y = \frac{-5}{4}x + 20 + 25$$

$$y' = \frac{-5}{4}x + 45$$

-1

(10 points) 8) Find the derivative of the function $y = (x)^{\cos x}$

$$y = x^{\cos x} \qquad \ln y = \cos x \ln x$$

$$\frac{1}{y}y' = -\sin x \ln x + \cos x \frac{1}{x}$$

$$y' = y\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

$$y' = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

$$h'(x) = \frac{g'(x)(w(x)) - w'(x)g(x)}{w^2(x)}$$

(6 Points) **9)** Suppose that $h(x) = \dfrac{g(x)}{w(x)}$ and $F(x) = g(g(x))$, where

$$F'(x) = g'(g(x)) \cdot g'(x)$$

$g''(2) = -1$

$w''(5) = 8$

$w(5) = 4,$

$g(5) = 2,$

$g'(5) = -1,$

$w'(5) = -2,$

$g'(2) = -5$

a) Find $h'(5)$

$$h'(5) = \frac{-1(4) - -2(2)}{4^2} = \boxed{0}$$

b) Find $F'(5)$.

$$F'(5) = g'(2)\, g'(5)$$
$$= (-5)(-1)$$
$$\boxed{= 5}$$

(5 Points) 10) Find all values of x so that the graph of $f(x) = \sqrt{3}\, x - 2\cos x$ will have a horizontal tangent?

$$f'(x) = \sqrt{3} + 2\sin x$$

$$\sqrt{3} + 2\sin x = 0$$
$$\sin x = -\frac{\sqrt{3}}{2}$$

$$\boxed{x = \frac{4\pi}{3} + 2n\pi \qquad x = \frac{5\pi}{3} + 2n\pi}$$

(10 Points)

11) Based on ~~the above~~ $S = f(t) = t^3 - 6t^2 + 9t \qquad t \geq 0$

where "t" is measured in seconds and "S" is in meters.

a) Find the total distance traveled by the particle during the **seven** seconds? (5 points)

$$S'(t) = 3t^2 - 12t + 9 \longrightarrow 3(t^2 - 4t + 3) = 0$$
$$3(t-3)(t-1) = 0$$
$t = 3$ seconds
$t = 1$ seconds

$$a(t) = 6t - 12$$

$$S_{0-1} + S_{1-3} + S_{3-7} = |4-0| + |0-4| + |112-0| = 112 + 8 = \boxed{120 \text{ meters}}$$

b) When is the particle slowing down? (2.5 points)

$\boxed{0 \leq t < 1 \text{ and } 2 < t < 3}$ seconds

c) When is the particle moving backward? (2.5 points)

when $v(t) < 0$

$\boxed{1 < t < 3}$ seconds

$M(x) = 2x + x^{\frac{3}{2}}$

(4 Points) 12) The mass of part of a wire is $x(2+\sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 16$ meters. (Hint: linear density $\rho = \frac{dm}{dx}$).

$\rho = \frac{dm}{dx} = 2 + \frac{3}{2}x^{\frac{1}{2}} \Rightarrow \rho = 2 + \frac{3}{2}(16)^{\frac{1}{2}}$

$= 2 + \frac{3}{2}(4) = \boxed{8 \frac{\text{Kg}}{\text{meter}}}$

(10 Points) 13) Find the equation of the tangent line to the parametric curve $x = t^2 + 3, \quad y = 2t^3 - t$ at the point corresponding to $t = 2$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 1}{2t + 0}\Big|_{t=2} = \frac{23}{4}$ $\quad x = 7, \; y = 14$

$y - 14 = \frac{23}{4}(x - 7)$

$\boxed{y = \frac{23}{4}x - \frac{105}{4}}$

14a) Find the linearization of $f(x) = \sqrt{x+3}$ at a = 1. and $f(1) = 2$
State the corresponding linear approximation. **(6 points)**

$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} \quad f'(1) = \frac{1}{2}(1+3)^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x - 1)$

$y = \frac{1}{4}x - \frac{1}{4} + 2 = \boxed{\frac{1}{4}x + \frac{7}{4}}$

$\sqrt{x+3} \approx \frac{1}{4}x + \frac{7}{4}$

14b) Use the above to give an approximate value for $\sqrt{3.98}$. **(3 points)**

$\sqrt{3.98} = \sqrt{x+3} \Rightarrow x = 0.98 \longrightarrow \frac{1}{4}(0.98) + \frac{7}{4} = \boxed{1.995}$

Extra Credits (3 points) 15) Find the equation of the tangent to the circle $x^2 = 100 - y^2$ at the point (6 , 8).

$$2x = 0 - 2yy' \qquad y' = \frac{-2x}{2y} = \frac{-x}{y}$$

$$y' = \frac{-6}{8} = \frac{-3}{4}$$

$$y - 8 = \frac{-3}{4}(x - 6)$$

$$y = \frac{-3}{4}x + \frac{18}{4} + 8 = \frac{-3}{4}x + \frac{18+32}{4} = \frac{-3}{2}x + \frac{50}{4}$$

$$y = \frac{-3}{4}x + \frac{25}{2}$$

Extra Credits (4 points) 16) Show that the following curves are orthogonal (i.e Perpendicular)

$2x^2 + y^2 = 3$

$x = y^2$

$$2x^2 + x = 3$$

$$2x^2 + x - 3 = 0$$

$$(2x+3)(x-1) = 0$$

$$\cancel{x = \frac{-3}{2}} \qquad x = 1$$

$$y = \pm 1$$

$$4x + 2yy' = 0 \qquad y' = \frac{-2x}{y}$$

$$1 = 2yy' \Rightarrow y' = \frac{1}{2y}$$

x	$y' = \frac{-2x}{y}$	$y' = \frac{1}{2y}$
(1,1)	−2	$\frac{1}{2}$
(1,−1)	2	$-\frac{1}{2}$

Negative Reciprocal slopes

Since the slopes are negative Reciprocal, then the curves are perpendicular

Extra Credits (3 points) 17) Show that the following curves are orthogonal.

$3x^2 + 2x - 3y^2 = 1$

$6xy + 2y = 0$

$$6x + 2 - 6yy' = 0 \Rightarrow y' = \frac{-6x-2}{-6y} = \frac{6x+2}{6y}$$

$$y' = \frac{3x+1}{3y}$$

$$6y + 6xy' + 2y' = 0$$

$$3y + 3xy' + y' = 0$$

$$y'(3x+1) = -3y \Rightarrow y' = \frac{-3y}{3x+1}$$

Since these slopes are negative Reciprocal of each other, these two curves are orthogonal.