Calculus I **Test II** **Professor: Fred Katiraie** **Spring 2006**

Name: Key

Total Possible Points = 140

Please Show Your Work for Full Credit, Include Units Whenever Possible.
Justify all your answers

1) (20 Points) Let f be a one-to-one function whose inverse function is given by the formula: $f^{-1}(x) = x^5 - 3x^3 + 5x + 2$

a) Compute $f^{-1}(-1) = (-1)^5 - 3(-1)^3 + 5(-1) + 2 \boxed{= -1}$

b) Compute $f(1)$ $\quad 1 = x^5 - 3x^3 + 5x + 2$

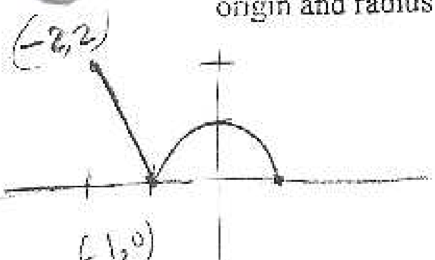
$x = -0.205 \Longrightarrow \boxed{f(1) = -0.205}$

c) Compute the value of x such that $f(x) = 1$

since $f^{-1}(1) = 5 \Longrightarrow \boxed{x = 5}$

d) Compute the value of y such that $f^{-1}(y) = 1$ $\quad 1 = y^5 - 3y^3 + 5y + 2 \Longrightarrow \boxed{y = -0.205}$

2) Find an expression for the function whose graph consists of the line segment from the point (-2,2) to the point (-1,0) together with the top half of the circle with the center at the origin and radius 1. (10 points)



$$f(x) = \begin{cases} -2x - 2 & x < -1 \\ \sqrt{1 - x^2} & -1 \leq x \leq 1 \end{cases}$$

$m = \dfrac{0-2}{-1--2} = \dfrac{-2}{1} = -2$

$y = mx + b$

$0 = -2(-1) + b \Rightarrow \boxed{b = -2}$

(10 Points) 3) A small-appliance manufacturer finds that it costs $9000 to produce 1000 toaster ovens a week and $12000 to produce 1500 toaster ovens a week.

(1000, 9000)
(1500, 12000)

a) Express the cost as a function of the number of the toaster ovens produced, assuming that it is linear.

$m = \dfrac{12000 - 9000}{500} = 6$

$C(x) = 6x + 3000$

$9000 = 6(1000) + b$

$3000 = b$

b) What is the slope of the graph and what does it represent?

$m = 6$ i.e each additional toaster costs $6 to produce

c) What is the y-intercept of the graph and what does it represent?

$y_{int} = (0, 3000)$ the fixed cost is $3000

4) A ball is thrown into the air with a velocity of 40 feet per second, its height in feet after t seconds is given by $y = 40t - 16t^2$ (4 Pts Each)

a) Find the average velocity for the time period beginning when t = 1 and lasting

t	$y = 40t - 16t^2$
1.00	24
1.05	24.36
1.01	24.0784

i) 0.05 s $\Longrightarrow$ Avg Velocity $= \dfrac{24.36 - 24}{0.05} = 7.2$ ft/sec

j) 0.01 s $\Longrightarrow$ Avg Velocity $= \dfrac{24.0784 - 24}{0.01} = 7.84$ ft/sec

b) Find the instantaneous velocity when t = 1 (2 Pts)

$$v(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{40(1+h) - 16(1+h)^2 - 24}{h} = \lim_{h\to 0} \frac{\cancel{40} + 40h - \cancel{16} - 32h - 16h^2 - \cancel{24}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(8 - 16h)}{\cancel{h}} = \boxed{8 \text{ ft/sec}}$$

5) Given $f(x) = \sqrt{1+2x}$

a) Find the domain of $f(x)$ (2 pts)

$1 + 2x \geq 0 \qquad 2x \geq -1 \qquad \boxed{x \geq -1/2}$

b) Use the definition of the a derivative to find $f'(x)$ (10 Pts)

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} = \lim_{h\to 0} \frac{\sqrt{1+2x+2h} - \sqrt{1+2x}}{h} \cdot \frac{\sqrt{1+2x+2h} + \sqrt{1+2x}}{\sqrt{1+2x+2h} + \sqrt{1+2x}}$$

$$= \lim_{h\to 0} \frac{(\cancel{1} + \cancel{2x} + 2h) - (\cancel{1} + \cancel{2x})}{h(\sqrt{1+2x+2h} + \sqrt{1+2x})} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{1+2x+2h} + \sqrt{1+2x})} =$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{1+2x+2h} + \sqrt{1+2x}} = \frac{\cancel{2}}{\cancel{2}\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$$

c) Find the domain of $f'(x)$ (3 pts)

$1 + 2x > 0 \Longrightarrow 2x > -1 \Longrightarrow \boxed{x > -1/2}$

6) Find the following limits algebraically: (Justify your Answer) (5 Pts Each)

a) $\lim_{t\to 0} \dfrac{1}{t} - \dfrac{1}{t^2+t} = \lim_{t\to 0} \dfrac{1}{t} - \dfrac{1}{t(t+1)}$

$$= \lim_{t\to 0} \frac{(t+1) - 1}{t(t+1)} = \lim_{t\to 0} \frac{t}{\cancel{t}(t+1)} = \lim_{t\to 0} \frac{1}{t+1} = \boxed{1}$$

6b) Find the following limits algebraically $\lim_{t\to 0}\frac{\sqrt{t^2+25}-5}{t^2}\cdot\frac{\sqrt{t^2+25}+5}{\sqrt{t^2+25}+5}$

$$= \lim_{t\to 0}\frac{(t^2+\cancel{25})-\cancel{25}}{t^2\left(\sqrt{t^2+25}+5\right)} = \lim_{t\to 0}\frac{1}{\sqrt{t^2+25}+5} = \boxed{\frac{1}{10}}$$

7) ~~Find~~ Show the following limit: $\lim_{x\to 0} x^4\cos\left(\frac{1}{x^2}\right)=0$

(Hint: Use the Squeeze Theorem) (5 Pts)

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le +1$$

$$x^4 \le x^4\cos\left(\frac{1}{x^2}\right) \le x^4$$

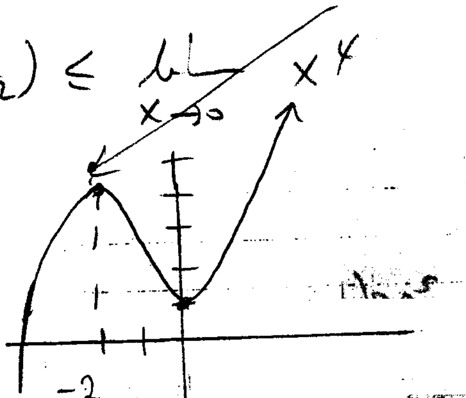
$$\lim_{x\to 0} x^4 \le \underbrace{\lim_{x\to 0} x^4\cos\left(\frac{1}{x^2}\right)} \le \lim_{x\to 0} x^4$$

0

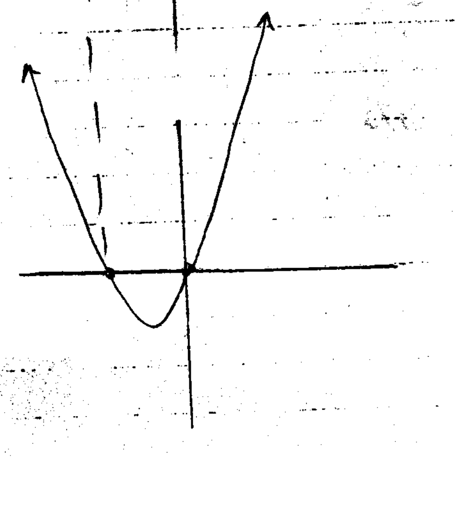
∴ Then By Squeeze theorem $\lim_{x\to 0} x^4\cos\left(\frac{1}{x^4}\right)=$

8) Given $f(x)=x^3+3x^2+1$

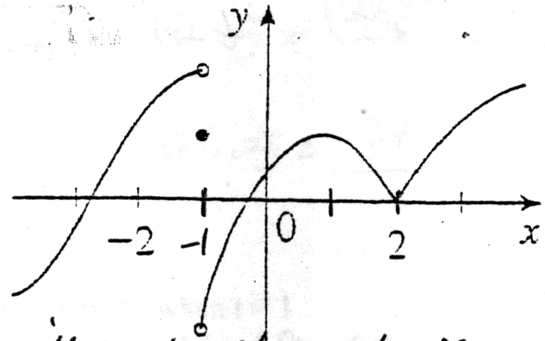
a) Graph the function (1 point)



b) Graph $f'(x)$ (4 points)



9) The graph of f (x) is shown below. State, with reasons, the numbers at which f (x) is not differentiable. (5 points)



f(X) is Not differentiable at $X = -1$ (B/c of Discontinuity)

and

f(x) is Not differentiable at $X = 2$ (B/c of Sharp Edge)

10) Given $f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

a) Is $f(x)$ continuous at $x = 0$? (must use definition of continuity) (5 points)

$\lim_{X \to 0} x^2 \cos \frac{1}{x} = 0$ and $f(0) = 0$

B/c

$-1 \leq \cos \frac{1}{x} \leq 1$

$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$

$\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$

Then by S.T $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$

since $\lim_{X \to 0} x^2 \cos\left(\frac{1}{x}\right) = f(0)$

the f(x) is continuous at 0

b) Is $f(x)$ differentiable at $x = 0$? (Justify your answer) (5 points)

since f(x) does not have a sharp edge, and is continuous at $x = 0$

and there are No vertical tangent at $x = 0$

then f(x) is differentiable at $x = 0$.

11) Sketch the graph of a function that satisfies all of the following conditions: (10 points)

$f'(-1) = f'(1) = 0$

$f'(x) < 0$ if $|x| < 1$

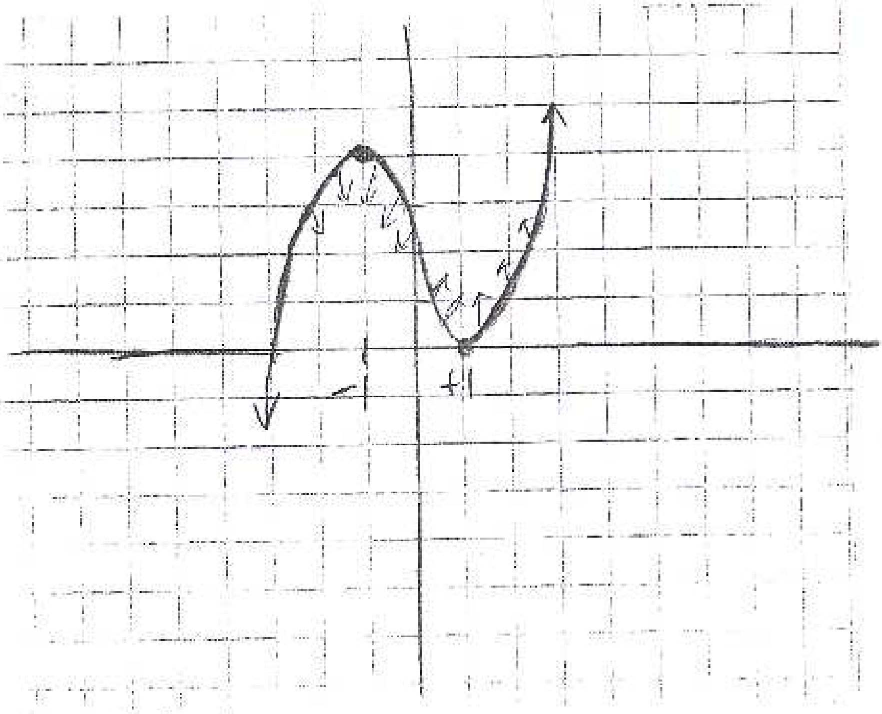
$f'(x) > 0$ if $|x| > 1$

$f(-1) = 4$

$f(1) = 0$

$f''(x) < 0$ if $x < 0$

$f''(x) > 0$ if $x > 0$

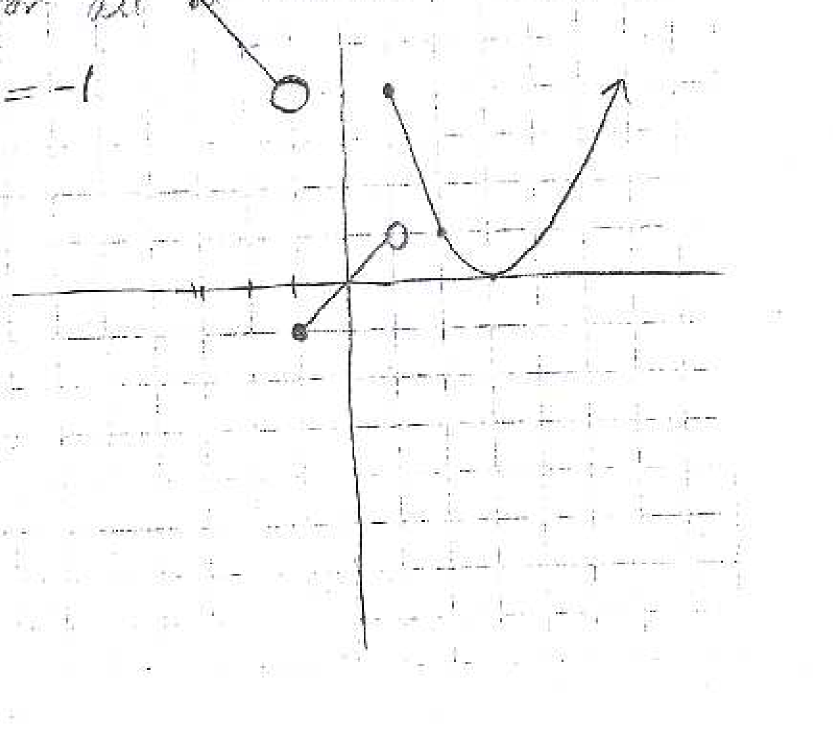


12) Sketch the graph of the following function

And use it to determine all the values of "a" for which $\lim_{x\to a} f(x)$ exists. (5 Pts)

$$f(x) = \begin{cases} 3-x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ (x-3)^2 & \text{if } x \ge 1 \end{cases}$$

So $\lim_{x \to a} f(x)$ exists for all a values in Real numbers except at $a = 1$ and $a = -1$



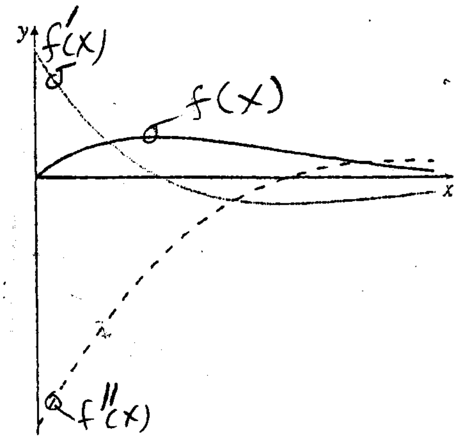
$x < -1$

x	y
-1	4
-2	5
-3	6

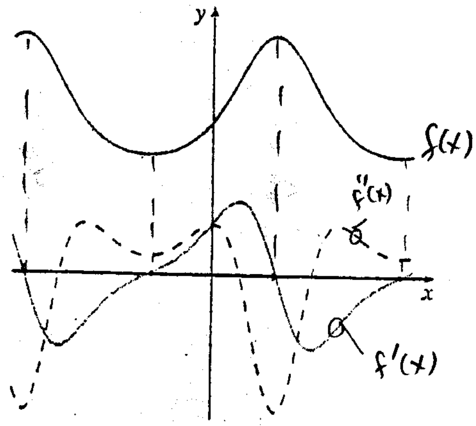
$x \ge 1$

x	$y = (x-3)^2$
1	4
2	1
3	0

13) Each figure below shows the graphs of a function, its first derivative, and its second derivative. Identify which is which. (10 points)



14) Each figure below shows the graphs of a function, its first derivative, and its second derivative. Identify which is which. (10 points)



15) Given $g(x) = x \ln x$ and $\begin{cases} g'(x) = \ln x + 1 \\ m = \ln e + 1 = 2 \end{cases}$

Find an equation of the tangent line at $x = e$ (5 points)

$y = e \ln e = e(1) = e$

$y - y_1 = m(x - x_1)$

$y - e = 2(x - e)$

$y = 2x - 2e + e \Rightarrow \boxed{y = 2x - e}$