Calculus I Test III Professor: Fred Katiraie Spring 2006

Name:______________________________ Score = $\frac{\quad}{140}$

Please Show Your Work for Full Credit, Include Units Whenever Possible.
Justify all your answers

1) Given $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

a) Is $f(x)$ continuous at $x=0$? (must use definition of continuity) Yes (5 points)

$\lim_{x\to 0} f(x) \overset{?}{=} f(0)$

$-1 \le \sin(\frac{1}{x}) \le 1$

$-x \le x\sin(\frac{1}{x}) \le x$

$\lim_{x\to 0} -x = 0$ and $\lim_{x\to 0} x = 0$,

By Squeeze theorem $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

$\lim_{x\to 0} x\sin(\frac{1}{x}) = f(0) = 0$

$\therefore f(x)$ is continuous at $x=0$

b) Is $f(x)$ differentiable at $x=0$? (Justify your answer) (5 points)

No, because f(x) has a sharp edge at $x=0$

$\therefore$ It is not differentiable at $x=0$

2) Sketch the graph of a function that satisfies all of the following conditions: (10 points)

$f'(-1)=f'(1)=0$

$(-1<x<1)$ $f'(x)<0$ if $|x|<1$

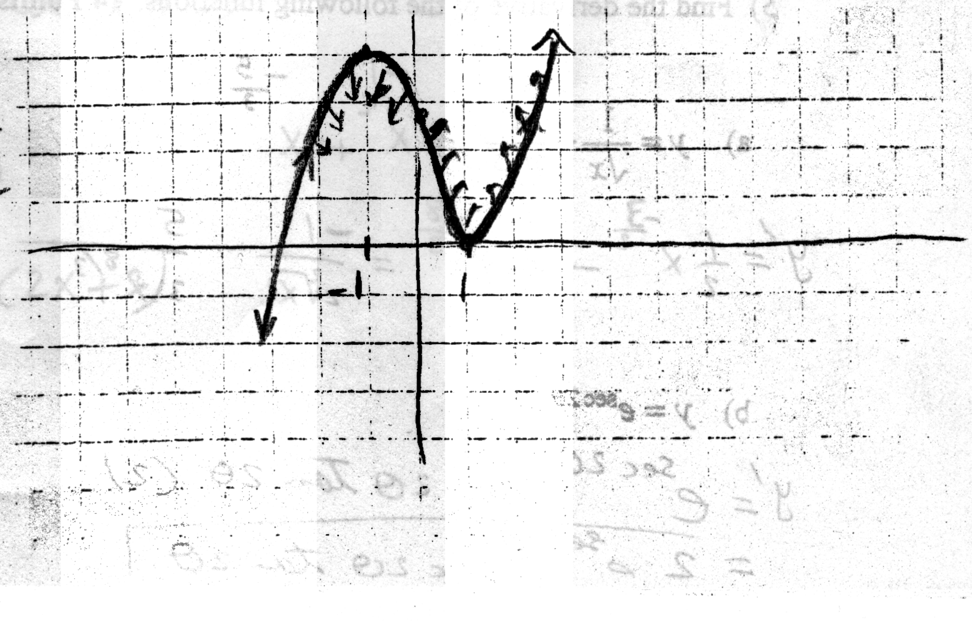
$(x<-1\ \ x>1)$ $f'(x)>0$ if $|x|>1$

$f(-1)=4$

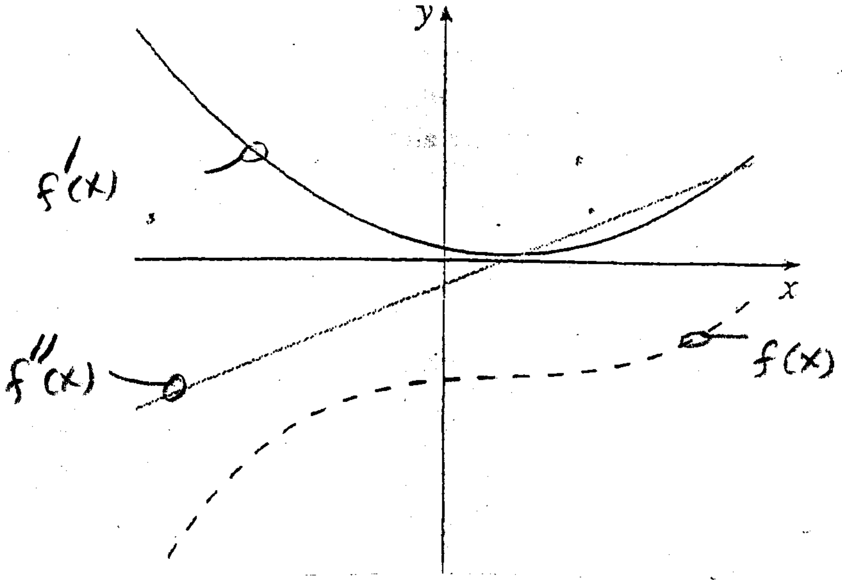
$f(1)=0$

$f''(x)<0$ if $x<0$

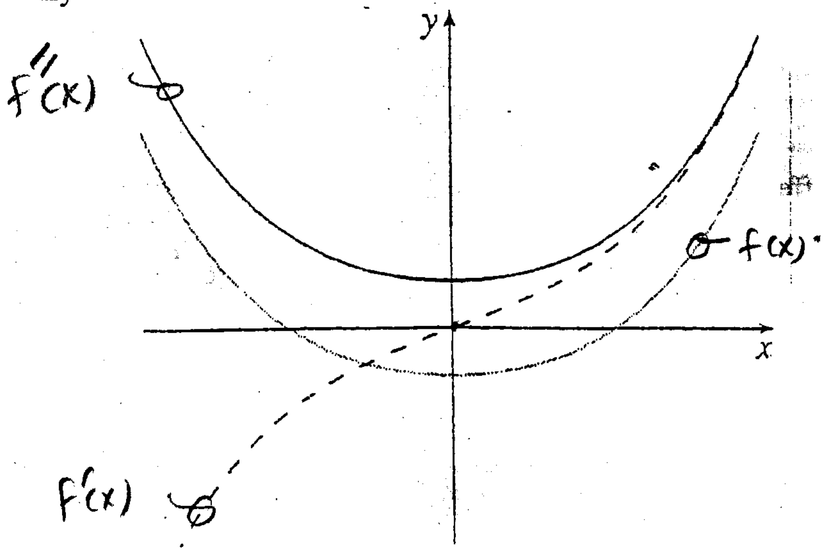
$f''(x)>0$ if $x>0$



3) Each figure below shows the graphs of a function, its first derivative, and its second derivative. Identify which is which. (10 points)



4) Each figure below shows the graphs of a function, its first derivative, and its second derivative. Identify which is which. (10 points)



5) Find the derivative of the following functions: **(4 Points each)**

(Do Not Simplify)

a) $y = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^5}} = x^{-\frac{1}{2}} + x^{-\frac{5}{3}}$

$$y' = \frac{1}{2} x^{-\frac{3}{2}} - \frac{5}{3} x^{-\frac{8}{3}} = \frac{-1}{2\sqrt[2]{x^3}} - \frac{5}{3x^{8/3}}$$

b) $y = e^{\sec 2\theta}$

$$y' = e^{\sec 2\theta} \sec 2\theta \tan 2\theta \,(2)$$

$$\boxed{= 2e^{\sec 2\theta} \sec 2\theta \tan 2\theta}$$

c) $y=\sin^5(3x^2+5x+1) = \left(\sin(3x^2+5x+1)\right)^5$

$$y' = 5\left(\sin(3x^2+5x+1)\right)^4\left(\cos(3x^2+5x+1)\right)(6x+5)$$

d) $y=\csc((7x)^3)$

$$y' = -\csc((7x)^3)\cot((7x)^3)\cdot 3(7x)^2(7)$$

e) $y=\sqrt{5x+\sqrt{x+5}} = \left(5x+(x+5)^{1/2}\right)^{1/2}$

$$y' = \tfrac{1}{2}\left(5x+(x+5)^{1/2}\right)^{-\frac{1}{2}}\left(5+\tfrac{1}{2}(x+5)^{-\frac{1}{2}}\right)$$

f) $y=\left(\dfrac{x-7}{x^2+1}\right)^9$

$$y' = 9\left(\frac{x-7}{x^2+1}\right)^8\cdot\left(\frac{1(x^2+1)-2x(x-7)}{(x^2+1)^2}\right) = 9\frac{(x-7)^8}{(x^2+1)^8}\left(\frac{-x^2+1+14x}{(x^2+1)^2}\right)$$

g) $y=17^{\cot\pi\theta}$

$$y' = 17^{\cot\pi\theta}\cdot\ln 17\cdot\left(-\csc^2(\pi\theta)\right)(\pi)$$

$$= -\pi(\ln 17)\,17^{\cot\pi\theta}\csc^2(\pi\theta)$$

h) $y=\sin^{-1}(x^2+2x+1)$

$$y' = \frac{1}{\sqrt{1-(x^2+2x+1)^2}}\cdot(2x+2)$$

6) Given $f(x)=e^{x^2-3x+2}$ (2 Points each)

a) Find the first derivative of $f(x)=e^{x^2-3x+2}$.

$f'(x)=\left(e^{x^2-3x+2}\right)\boxed{(2x-3)}$

b) Find the second derivative of $f(x)=e^{x^2-3x+2}$

$f''(x)=\left(e^{x^2-3x+2}\right)(2x-3)^2+\boxed{2e^{x^2-3x+2}}$

c) Evaluate the second derivative of f (x) at x = 2. (In other words, find $f''(2)$.)

$f''(2)=\left(e^{2^2-3(2)+2}\right)(2(2)-3)^2+2e^{2^2-3(2)+2}$

$=e^0(1)^2+2e^0 = 1+2 \boxed{=3}$

7) Given $h(x)=\sqrt{1-x}$ (4 points each)

$h(x)=(1-x)^{1/2}$ $h'(x)=\frac{1}{2}(1-x)^{-1/2}(-1) \Rightarrow \boxed{h'(0)=-\frac{1}{2}}$

a) Find a linearization of $h(x)=\sqrt{1-x}$ at $\boxed{a=0,\ y=\sqrt{1-0}=1}$

$=f'(a)(x-a)+f(a)$

$=-\frac{1}{2}(x-0)+1 = \boxed{-\frac{1}{2}x+1}$

b) Use your answer to estimate $\sqrt{0.99}$

$\sqrt{0.99}=\sqrt{1-x}$ $\Rightarrow \sqrt{0.99}=-\frac{1}{2}(0.01)+1 \boxed{=0.995}$

$0.99=1-x$

$0.99-1=-x \Rightarrow \boxed{x=+0.01}$

(8 Points) 8) Find the equation of the tangent line to the curve $\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=\frac{7}{12}$, at the point (9,16).

$x^{-\frac{1}{2}}+y^{-\frac{1}{2}}=\frac{7}{12}$

$-\frac{1}{2}x^{-\frac{3}{2}}+-\frac{1}{2}y^{-\frac{3}{2}}y'=0$ (Multiplying by -2)

$x^{-\frac{3}{2}}+y^{-\frac{3}{2}}y'=0$

$y'=\frac{-x^{-\frac{3}{2}}}{y^{-\frac{3}{2}}}=-\frac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}}$

(9,16)

$y'=-\frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}}}=-\frac{64}{27}$

$y-16=-\frac{64}{27}(x-9)$

$y=-\frac{64}{27}x+\frac{64}{3}+16$

$\boxed{y=-\frac{64}{27}x+\frac{112}{3}}$

(7 points) 9) Let $y = f(x)$ be implicitly defined as $x^{\sin y} = y^{\cos x}$

Compute y' in terms of x, and y. (Hint: Use Natural Logarithms)

$$\sin y \ln x = \cos x \ln y$$

$$\cos y\, y' \ln x + \sin y \frac{1}{x} = -\sin x \ln y + \cos x \frac{1}{y} y'$$

$$y'\left(\cos y \ln x - \cos x \frac{1}{y}\right) = -\sin x \ln y - \sin y \frac{1}{x}$$

$$y' = \frac{-\sin x \ln y - \frac{\sin y}{x}}{\cos y \ln x - \cos x \cdot \frac{1}{y}}$$

(6 Points) 10) Suppose that $h(x) = \dfrac{g(x)}{w(x)}$, and $F(x) = g(g(x))$, where

$g''(2) = -1$

$w''(5) = 8$

$w(5) = 4$

$g(5) = 2$

$g'(5) = -1$

$w'(5) = -2$

$g'(2) = -5$

a) Find $h'(5) = \dfrac{g'(5)w(5) - w'(5)g(5)}{[w(5)]^2}$

$$h'(5) = \frac{-1(4) - (-2)(2)}{(4)^2}$$

$$= \frac{-4+4}{16} = \boxed{0}$$

b) Find $F'(5)$.

$$F'(x) = g'(g(x))\, g'(x)$$

$$= g'(2)\, g'(5)$$

$$= (-5)(-1)$$

$$\boxed{= 5}$$

(6 Points) 11) Consider the circle $x^4 + y^4 = 1$.

a) At what point(s) is the slope of the tangent line equal to 1?

$$4x^3 + 4y^3 y' = 0 \Rightarrow y' = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3} = 1$$

$$\Rightarrow y^3 = -x^3 \Rightarrow y = -x$$

$$x^4 + (-x)^4 = 1 \Rightarrow 2x^4 = 1 \quad x = \pm\sqrt[4]{\tfrac{1}{2}}$$

$$\left(\pm\sqrt[4]{\tfrac{1}{2}}, \mp\sqrt[4]{\tfrac{1}{2}}\right)$$

b) At what point(s) is the slope of the tangent line equal to 0?

$$\frac{-x^3}{y^3} = 0 \Rightarrow \boxed{x = 0}$$

$$0 + y^4 = 1 \Rightarrow y = \pm 1$$

$(0,1)$

$(0,-1)$

Show All Your Work, **No Procedure = No Points**

(6 Points)

12) The position of a particle is given by the equation

$S = f(t) = t^3 - 6t^2 + 9t \qquad t \geq 0$,

where "t" is measured in seconds and "S" is in meters.

a) When is the particle at rest?

$V(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3)$
$= 3(t-3)(t-1)$

set $V(t) = 0$

$3(t-3)(t-1) = 0 \implies \boxed{t = 1 \text{ sec}} \quad \boxed{t = 3 \text{ sec}}$

b) When is the particle moving forward? Particle Moves forward when velocity is positive

$0 \leq t < 1$

$3 < t$ seconds

(6 Points)

13) Based on the above $S = f(t) = t^3 - 6t^2 + 9t \qquad t \geq 0$

where "t" is measured in seconds and "S" is in meters.

a) Find the total distance traveled by the particle during the first **five** seconds? (2 points)

$\left.\begin{array}{l} 0 \leq t < 1 \longrightarrow 4 \text{ meters} \\ 1 < t < 3 \longrightarrow 4 \text{ meters} \\ 3 < t < 5 \longrightarrow 20 \text{ meters} \end{array}\right\}$ total Distance $= 28$ meters

b) When is the particle speeding up? (2 points)

"a" and "velocity" have same signs

$1 < t < 2$ seconds

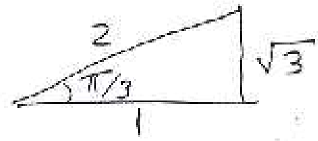
$3 < t$ seconds

c) When is the particle slowing down? (2 points)

"a" and "v" have different signs

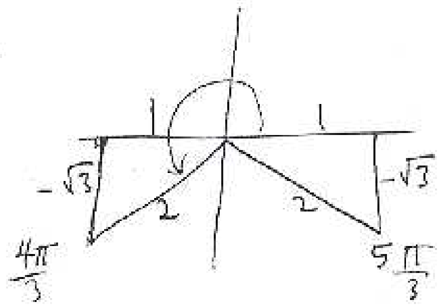
$0 \leq t < 1$

$2 < t < 3$



(7 Points) 14) Find all values of x so that the graph of $f(x) = \sqrt{3}\,x - 2\cos x$ will have a horizontal tangent?

$$f'(x) = \sqrt{3} + 2\sin x = 0$$



$$2\sin x = -\sqrt{3}$$

$$\sin x = -\frac{\sqrt{3}}{2}$$

$$\boxed{\begin{aligned} x &= \frac{4\pi}{3} + 2n\pi \\ x &= \frac{5\pi}{3} + 2n\pi \end{aligned}}$$

(7 Points) 15) Find the slope of the tangent line to the curve $x^3 + y^3 = 6xy$ at the point (3 , 3).

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)} = \frac{2y - x^2}{y^2 - 2x}$$

If $x = 3$ and $y = 3$ $\quad y' = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = \boxed{-1}$

EXTRA CREDIT PROBLEMS

(Extra Credit 4 Points) 16) Given the curve $y = e^x$

a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line

$x - y - 1 = 0$

$y = x - 1 \quad \boxed{m = 1}$

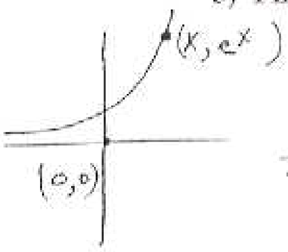
$y' = e^x = 1 \Longrightarrow \cancel{\ln} \ln e^x = \ln 1 \Longrightarrow \boxed{x = 0} \; \boxed{y = e^0 = 1}$ point

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0)$

$\boxed{y = x + 1}$ eqn of tangent line

b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin



$m = \dfrac{e^x - 0}{x - 0} = \dfrac{e^x}{x}$ Also $y' = e^x \Longrightarrow \boxed{m = e^1 = e}$

then $\dfrac{e^x}{x} = e^x \Longrightarrow \boxed{x = 1}, \boxed{y = e^1 = e} \Longrightarrow y - e = e(x - 1)$

$\boxed{y = ex - \cancel{e} + \cancel{e} = ex}$

(Extra Credit 2 points) 17) Consider the curve given by $x = t^2 + 3, \quad y = 2t^3 - t$.

Find $\dfrac{dy}{dx}$ at the point corresponding to $t = 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 1}{2t}\Bigg|_{t=2} = \frac{6(2)^2 - 1}{2(2)} = \boxed{\frac{23}{4}}$$

(Extra Credit 2 points) 18) Show that the following curves are orthogonal (i.e Perpendicular)

$3x^2 + 2x - 3y^2 = 1$
$6xy + 2y = 0$

$6x + 2 - 6yy' = 0$

$6yy' = 6x + 2$

$y' = \dfrac{6x + 2}{6y} = \dfrac{\cancel{2}(3x + 1)}{\cancel{2} \cdot 3y} = \dfrac{3x + 1}{3y}$

$6xy + 2y = 0$

$6y + 6xy' + 2y' = 0$

$y'(6x + 2) = -6y$

$y' = \dfrac{-6y}{6x + 2} = \dfrac{-3y}{3x + 1}$

Since the y's are negative reciprocal of each other these curves are orthogonal.