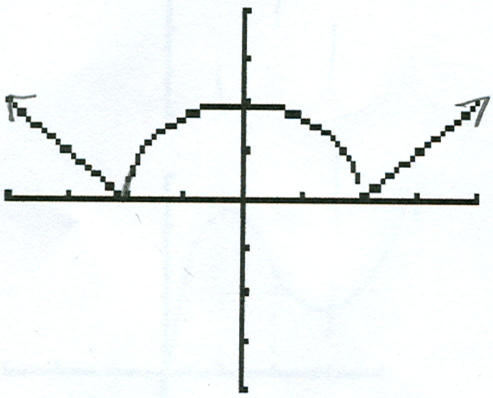
Professor Katiraie Calculus I Fall 2006 Test III (chapters 1, 2, and 3)

Name:____________________ Total Possible Points = 140 (Plus 10 pts Extra Credits ☺)

Show All Your Work, No Procedure = No Points

(10 Points) 1) Find a formula that describes the following function:



$$f(x) = \begin{cases} -x-2 & x<-2 \\ \sqrt{4-x^2} & -2 \le x \le 2 \\ x-2 & x>2 \end{cases}$$

2) Use the intermediate Value Theorem to show that there is a root of the equation $x^3+2x^2-42=0$ on the interval (0,3). let $f(x)=x^3+2x^2-42$ **(5 Points)**

Note that $f(x)$ is continuous over $[0,3]$
and $f(0)=-42$ and $f(3)=3$ and $f(0)\neq f(3)$
then by IVT $\exists\, c \in (0,3)$ s.t $f(c)=0$

3) Find the following limits **Algebraically** **(5 Points)**

a) $\lim_{x\to\infty}\frac{1-2x^2}{x^2+x}$ $\frac{\div x^2}{\div x^2}$

$$\lim_{x\to\infty}\frac{\frac{1}{x^2}-\frac{2x^2}{x^2}}{\frac{x^2}{x^2}+\frac{x}{x^2}} = \frac{-2}{1} = \boxed{-2}$$

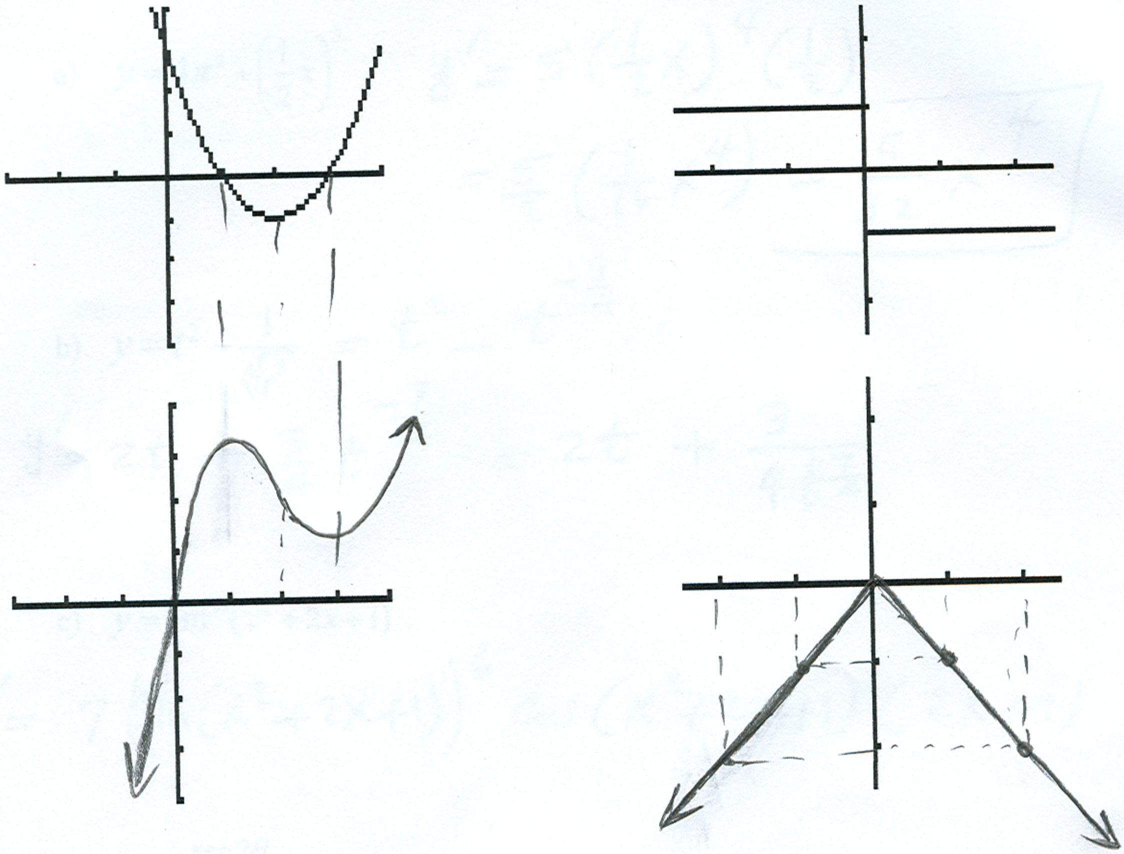
b) $\lim_{x\to\infty}(\sqrt{x^2+2x}-x)\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}+2x-\cancel{x^2}}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty}\frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}+\frac{x}{x}}$$

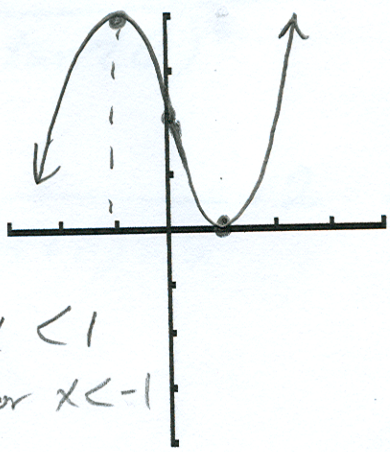
$$=\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=\boxed{1}$$

(10 Points)

4) Given the graphs of $y = f'(x)$, sketch the graphs of $y = f\ (x)$



5) Sketch the graph of a function that satisfies all of the following conditions: **(10 points)**



$f'(-1) = f'(1) = 0$

$f'(x) < 0$ if $|x| < 1$ $\quad -1 < x < 1$

$f'(x) > 0$ if $|x| > 1$ $\quad 1 < x$ or $x < -1$

$f(-1) = 4$ ✓

$f(1) = 0$ ✓

$f''(x) < 0$ if $x < 0$

$f''(x) > 0$ if $x > 0$

6) Find the derivative of the following functions: **(4 Points each)**

(Do Not Simplify)

a) $y = 4\pi^2 + \left(\frac{1}{2}x\right)^5$

$y' = 5\left(\frac{1}{2}x\right)^4\left(\frac{1}{2}\right)$

$= \frac{5}{2}\left(\frac{1}{16}x^4\right) = \boxed{\frac{5}{32}x^4}$

b) $y = t^2 - \frac{1}{\sqrt[4]{t^3}} = t^2 - t^{-\frac{3}{4}}$

$y' = 2t + \frac{3}{4}t^{-\frac{7}{4}} = 2t + \frac{3}{4t^{\frac{7}{4}}}$

c) $y = \sin^7(x^2 + 2x + 1)$

$y' = 7\left(\sin(x^2+2x+1)\right)^6 \cos(x^2+2x+1)(2x+2)$

d) $y = \frac{\sec 2\theta}{1 + \tan 2\theta}$

$y' = \frac{2\sec 2\theta \tan 2\theta (1+\tan 2\theta) - 2\sec^2 2\theta \sec(2\theta)}{(1+\tan 2\theta)^2}$

e) $y = \log_5(1 + 2x)$

$y' = \frac{1}{(1+2x)\ln 5} \cdot 2 = \frac{2}{(1+2x)\ln 5}$

f) $y = \sin^{-1}(x^2 + 2x + 1)$

$y' = \frac{1}{\sqrt{1-(x^2+2x+1)^2}} \cdot (2x+2)$

g) $y = 17^{\cot \pi\theta}$

$y' = 17^{\cot \pi\theta} \ln 17 \cdot (-\csc^2 \pi\theta)\, \pi$

8)(2 Points each) a) Find the first derivative of $g(x) = x\sin(x)$.

$\boxed{g'(x) = 1\sin x + x\cos x}$

b) Find the second derivative of $g(x) = x\sin(x)$.

$g''(x) = \cos x + 1\cos x - x\sin x$

$= \boxed{2\cos x - x\sin x}$

c) Evaluate the second derivative of g(x) at $x = \frac{\pi}{6}$. (In other words, find $g''\left(\frac{\pi}{6}\right)$

$g''\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} - \frac{\pi}{6}\sin\frac{\pi}{6}$

$= 2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{6}\left(\frac{1}{2}\right) = \boxed{\sqrt{3} - \frac{\pi}{12}}$

9) Given $h(x) = \sqrt[3]{1+3x}$ (4 points each)

$h'(x) = \frac{1}{\cancel{3}}(1+3x)^{-\frac{2}{3}}(\cancel{3})$ $h'(0)=1$ $h(0)=1$

a) Find a linearization of $h(x) = \sqrt[3]{1+3x}$ at $a = 0$

$y = h'(a)(x-a) + h(a)$

$y = 1(x-0) + 1 = \boxed{x+1}$

b) Use your answer to estimate $\sqrt[3]{1.03}$ $\approx 0.01 + 1 = \boxed{1.01}$

$\sqrt[3]{1.03} = \sqrt[3]{1+3x}$

$1.03 = 1+3x$

$0.03 = 3x \Rightarrow x = 0.01$

(8 Points) 10) Find the equation of the tangent line to the curve $\sqrt{x} + \sqrt{y} = 9$ at the point (16 , 25).

$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}y' = 0$

$x^{-\frac{1}{2}} + y^{-\frac{1}{2}}y' = 0$

$y' = \frac{-x^{-\frac{1}{2}}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}} = -\frac{\sqrt{25}}{\sqrt{16}} = \frac{-5}{4}$

$y - 25 = \frac{-5}{4}(x - 16)$

$\boxed{y = \frac{-5}{4}x + 45}$

(7 points) **11)** Find the derivative of the function $y=(x)^{\cos x}$

Compute y' in terms of x (Hint: Use Natural Logarithms)

$$\ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \ln x$$

$$\frac{1}{y}y' = -\sin x \ln x + \cos x \frac{1}{x}$$; Now Multiply both sides by y

$$y' = \left(-\sin x \ln x + \frac{\cos x}{x}\right)\cdot\left(x^{\cos x}\right)$$

(8 Points) **12)**

Suppose that $h(x)=f(x)g(x)$, and $F(x)=f(g(x))$, where

$f'(5)=11$

$w''(5)=8$

$f(2)=3$

$g(2)=5$

$g'(2)=4$

$w'(5)=-2$

$f'(2)=-2$

a) Find $F'(2) = f'(g(2))\,g'(2)$

$$= f'(5)(4)$$

$$= (11)(4) = \boxed{44}$$

b) Find $h'(2)$.

$$= f'(2)g(2) + g'(2)f(2)$$

$$= (-2)(5) + (4)(3)$$

$$= -10 + 12 = \boxed{2}$$

(4 Points) 13) Given the following ellipse $x^2+2y^2=1$.

a) At what point(s) is the slope of the tangent line equal to 1?

$$2x + 4yy' = 0 \qquad y' = -\frac{2x}{4y} = \frac{-x}{2y} = 1 \Rightarrow y = \frac{-x}{2}$$

$$x^2 + 2\left(\frac{-x}{2}\right)^2 = 1$$

$$x^2 + 2\frac{x^2}{4} = 1 \Longrightarrow \frac{3}{2}x^2 = 1 \Longrightarrow \left(x = \pm\sqrt{\frac{2}{3}},\ y = \mp\frac{\sqrt{\frac{2}{3}}}{2}\right)$$

b) At what point(s) is the slope of the tangent line equal to 0?

(4 Points)

$$y' = \frac{-x}{2y} = 0 \Longrightarrow x = 0$$

$x^2+2y^2=1$ and $2y^2=1$

$$y^2 = \frac{1}{2} \qquad y = \pm\sqrt{\frac{1}{2}}$$

$$\left(0, +\sqrt{\tfrac{1}{2}}\right)\ \left(0, -\sqrt{\tfrac{1}{2}}\right)$$

14) The mass of part of a wire is $m = x(2+\sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 16$ meters. (Hint: linear density $\rho = \frac{dm}{dx}$). **(8 Points)**

$$\rho = \frac{dm}{dx} = 1(2+\sqrt{x}) + x\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)$$

$$\rho(16) = (2+\sqrt{16}) + 16\left(\tfrac{1}{2}16^{-\frac{1}{2}}\right) = \boxed{8\ \frac{\text{kg}}{\text{m}}}$$

(9 Points) 15) Let $y = e^{\frac{x}{5}}$.

$$y' = \frac{dy}{dx} = e^{\frac{x}{5}} \cdot \frac{1}{5}$$

a) Find the differential dy.

$$\boxed{dy = e^{\frac{x}{5}} \cdot \frac{1}{5} \cdot dx}$$

b) Evaluate dy if $x = 0$, and $dx = 0.3$

$$dy = e^{0}\,\frac{1}{5}(0.3) = \frac{0.3}{5} = \boxed{0.06}$$

c) Evaluate Δy if $x = 0$, and $dx = 0.3$ (Hint: $\Delta y = f(x+\Delta x) - f(x)$)

$$\Delta y = f(0+0.3) - f(0) = e^{\frac{0.3}{5}} - e^{\frac{0}{5}} = \boxed{0.0618}$$

(8 Points) 16) Find the **equation of the tangent line** to the parametric curve $x = t^2 + 3$, $y = 2t^3 - t$ at the point corresponding to $t = 2$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2-1}{2t} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{t=2} = \frac{6(2)^2-1}{2(2)} = \frac{23}{4}$$

$$y = 2(2)^3 - 2 = 16 - 2 = 14$$

$$x = 2^2 + 3 = 7$$

$$y - 14 = \frac{23}{4}(x-7)$$

$$y = \frac{23}{4}x - 26.25 \qquad \boxed{y = \frac{23}{4}x - \frac{105}{4}}$$

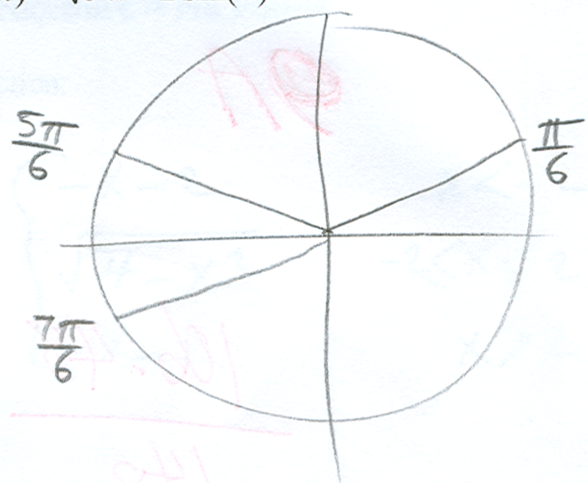
EXTRA CREDIT PROBLEMS

(5 Points) 16) Find all values of x so that the graph of $g(x)=\sqrt{3}\,x+2\sin(x)$ will have a horizontal tangent?

$g'(x)=\sqrt{3}+2\cos x=0$

$2\cos x=-\sqrt{3}$

$\cos x=-\frac{\sqrt{3}}{2}$



$$\boxed{\begin{aligned} x&=\frac{5\pi}{6}+2n\pi \\ x&=\frac{7\pi}{6}+2n\pi \end{aligned}}$$

(5 Points) 17) Show that the following curves are orthogonal (i.e Perpendicular)

$2x^2+y^2=3$

$x=y^2$

Intersection $\Rightarrow 2x^2+x=3 \Rightarrow 2x^2+x-3=0$

$(2x+3)(x-1)=0$

$\cancel{x=-\frac{3}{2}}$ $\quad x=1,\ y=\pm 1$

$4x+2yy'=0 \Rightarrow y'=\frac{-4x}{2y}=\boxed{\frac{-2x}{y}}$

$1=2yy' \Rightarrow y'=\frac{1}{2y}$

if $x=1$ & $y=1$

$$\boxed{\begin{aligned} y'&=\frac{-2(1)}{1}=-2 \\ y'&=\frac{1}{2} \end{aligned}}$$

If $x=1$ and $y=-1$

$$\boxed{\begin{aligned} y'&=\frac{-2(1)}{-1}=2 \\ y'&=\frac{1}{2(-1)}=-\frac{1}{2} \end{aligned}}$$

slopes are Negative Reciprocal