Professor Katiraie Calculus I Spring 2008 Test II Form C (chapters 1 and 2)

Name: Solution Total Possible Points = 140
(Plus 10 pts Extra Credit ☺)

1) (8 pts) Given the function: $f(x)=\dfrac{1}{x-2}$ Find the following $\dfrac{f(x+h)-f(x)}{h}$
(Clearly state each step of the process).

$f(x+h)=\dfrac{1}{x+h-2}$; $f(x+h)-f(x)=\dfrac{1}{x+h-2}-\dfrac{1}{x-2}=\dfrac{x-2-(x+h-2)}{(x+h-2)(x-2)}$

$f(x+h)-f(x)=\dfrac{-h}{(x+h-2)(x-2)}$

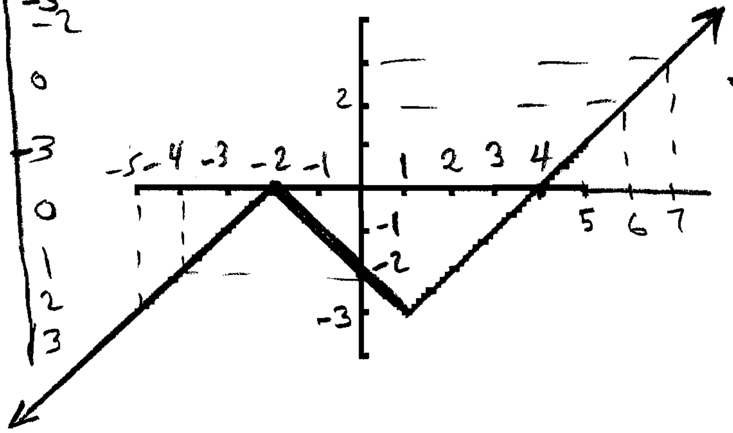
$$\frac{f(x+h)-f(x)}{\frac{h}{1}}=\frac{-\cancel{h}}{(x+h-2)(x-2)}\cdot\frac{1}{\cancel{h}}=\boxed{\frac{-1}{(x+h-2)(x-2)}}$$

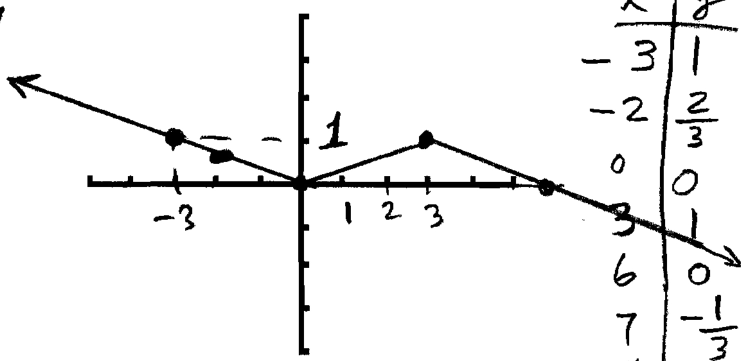
2) The graph of $y=f(x)$ is given below; Sketch a graph of $y=-\frac{1}{3}f(x-2)$ (8 points)

shift Right 2 units & Multiply the y by $\frac{-1}{3}$

old

x	y
-5	-3
-4	-2
-2	0
1	-3
4	0
5	1
6	2
7	3

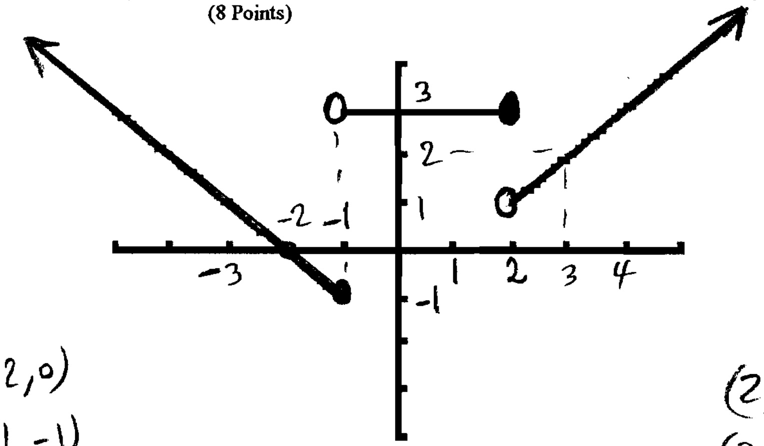




New

x	y
-3	1
-2	$\frac{2}{3}$
0	0
3	1
6	0
7	$-\frac{1}{3}$
8	$-\frac{2}{3}$
9	-1

3) Find a formula that describes the following function: (8 Points)



Answer:

$$\boxed{f(x)=\begin{cases}-1x-2 & x\le -1\\ 3 & -1<x\le 2\\ 1x-1 & 2<x\end{cases}}$$

$(-2,0)$
$(-1,-1)$
$m=\dfrac{-1-0}{-1+2}=\dfrac{-1}{1}=-1$

$(2,1)$
$(3,2)$ $m=\dfrac{2-1}{3-2}=1$

4) Solve the following algebraically: (6 points)

a) $\left(\frac{1}{3}\right)^{2-x} = 27$

$\left(3^{-1}\right)^{2-x} = 3^{3}$

$3^{-2+x} = 3^{3}$; $-2+x = 3$

$x = 5$

b) $e^{x^2} \bullet \frac{1}{e^{-6}} = \left(e^{-5x}\right)$

$e^{x^2+6} = e^{-5x}$

$x^2+5x+6=0$

$(x+3)(x+2)=0$

$x=-3$ $\quad$ $x=-2$

c) If $3^x = \frac{1}{10}$, what does 3^{-4x} equal?

$3^{-4x} = \left(3^x\right)^{-4} = \left(\frac{1}{10}\right)^{-4} = 10000$

5. Find the inverse of the following functions. (10 points)
(Must Show All the Appropriate Steps)

a) $y=\sqrt[4]{x+7}-6$	b) $f(x)=-\frac{1}{6}\ln(7x)$
Swap x and y $x = \sqrt[4]{y+7} - 6$ solve for y $x+6 = \sqrt[4]{y+7}$ $(x+6)^4 = y+7$ $(x+6)^4 - 7 = y$	$y = -\frac{1}{6}\ln(7x)$ • Swap x and y $x = -\frac{1}{6}\ln(7y)$ • Solve for y $-6x = \ln(7y)$ $e^{-6x} = 7y$ $y = \frac{e^{-6x}}{7}$

6) If an arrow is shot upward on the planet X with a velocity of 60 m/s, its height in meters after t seconds is given by $h(t) = 60t - 2t^2$ (10 Points)

a) Find the average velocity over the given time intervals:

t	$h(t)$
2	112
2.5	137.5
2.1	117.18
2.01	112.5198
2.001	112.052

i) [2 , 2.5] $\bar{V} = \dfrac{137.5 - 112}{2.5 - 2} = 51$ m/sec

j) [2 , 2.1] $\bar{V} = \dfrac{117.18 - 112}{2.1 - 2} = 51.8$ m/sec

k) [2 , 2.01] $\bar{V} = \dfrac{112.5198 - 112}{2.01 - 2} = 51.98$ m/sec

l) [2 , 2.001] $\bar{V} = \dfrac{112.052 - 112}{2.001 - 2} = 52$ m/sec

b) Find the instantaneous velocity after two seconds.

$V = 52$ m/sec

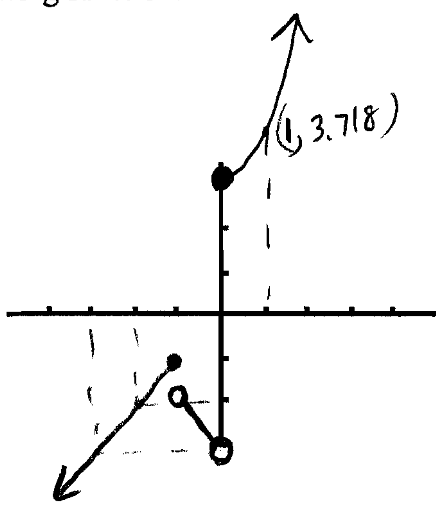
7a) Sketch the graph of the following function:

$$f(x) = \begin{cases} x & x \le -1 \\ -x-3 & -1 < x < 0 \\ e^x + 2 & x \ge 0 \end{cases}$$

(5 Points)

x	$y = x$
-1	-1
-2	-2
-3	-3

x	y
-1	-2
0	-3



7b) Discuss **(with reasons)** where the function f(x) is **discontinuous** and why. (5 Points)

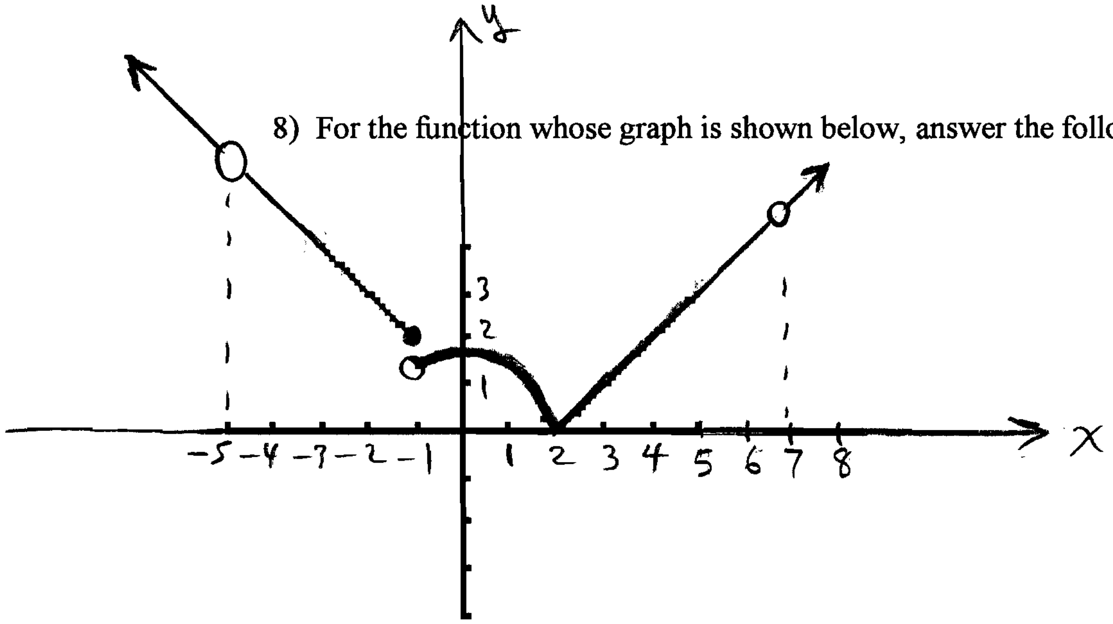
f(x) is discontinuous at $x = 0$ and $x = -1$

$\lim_{x \to -1} f(x) = DNE$ and $f(-1) = -1 \Rightarrow \lim_{x \to -1} f(x) \neq f(-1)$

$\lim_{x \to 0} f(x) = DNE$ and $f(0) = 3 \Rightarrow \lim_{x \to 0} f(x) \neq f(0)$

8) For the function whose graph is shown below, answer the following equations:

(9 Points)



a) At what number "a" $\lim_{x\to a} f(x)$ does **not** exist?

$\boxed{a=-1}$ B/c $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$

b) At what numbers "a" $\lim_{x\to a} f(x)$ exists, yet $f(x)$ is **not continuous**?

$\boxed{a=-5}$ and $\boxed{a=7}$ B/c $\lim_{x\to a} f(x) \neq f(a)$

c) At what numbers "a" $f(x)$ is continuous, but is **not differentiable**?

$a=2$

B/c $f(x)$ has a very sharp edge at $a=2$

9) $f(x) = \begin{cases} x^3+2 & x \le -2 \\ x^2+x+1 & -2 < x < 1 \\ x^4+3 & x \ge 1 \end{cases}$ (10 Points)

Find the following limits (give reasons, if the limit does not exist)

a) $\lim_{x\to 1} f(x) = \text{DNE}$

B/c $\lim_{x\to 1^-} f(x) = 3$ and $\lim_{x\to 1^+} f(x) = 4$

b) $\lim_{x\to -1} f(x) = (-1)^2 + (-1) + 1$

$= \boxed{1}$

c) $\lim_{x\to -2^+} f(x)$

$= (-2)^2 + (-2) + 1$

$= 4 - 2 + 1 = \boxed{3}$

d) $\lim_{x\to -3} f(x) = (-3)^3 + 2$

$= -27 + 2 = \boxed{-25}$

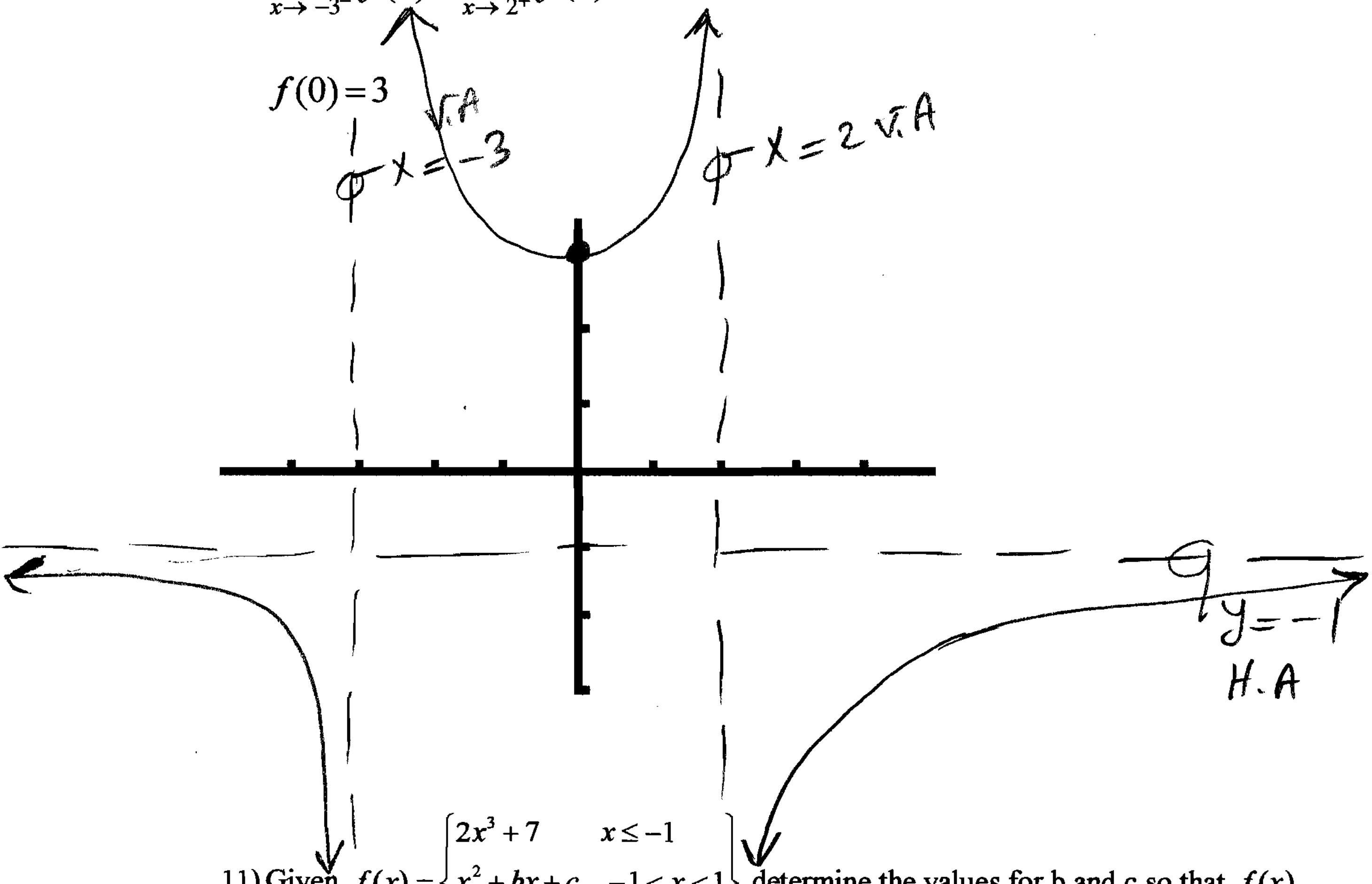
10) Given the following information about the limits, sketch a graph which could be the graph of y = $f(x)$. **Label all horizontal and vertical asymptote(s).** (9 Points)

$$\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = -1$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to 2^-} f(x) = \infty$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to 2^+} f(x) = -\infty$$

$f(0)=3$



11) Given $f(x)=\begin{cases} 2x^3+7 & x\le -1 \\ x^2+bx+c & -1<x<1 \\ x^4-10 & x\ge 1 \end{cases}$ determine the values for b and c so that $f(x)$ is continuous everywhere. (10 Points)

$2(-1)^3+7 = (-1)^2+b(-1)+c \Rightarrow \begin{cases} 1-b+c=5 \\ 1+b+c=-9 \end{cases}$

$1-10 = 1+b+c$

$2+2c=-4$

$2c=-6$

$1+b-3=-9$

$b=-9+2 \Rightarrow \boxed{b=-7}$

$\boxed{c=-3}$

12) Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points (2 , 5) and (4 , -5). Find the following: (12 Points)

a) Find $f'(3)$ $= \frac{-5-5}{4-2} = \frac{-10}{2} = -5$

b) Find $f(3)$ $= -5(3)+15 = 0$

c) Find an equation of the line tangent to f at $x = 3$

$y-5 = -5(x-2) \Rightarrow y = -5x+10+5 = -5x+15$

13) Find the following limits: (12 Points)

a) $\lim_{t\to 3} \frac{\sqrt{t+6}-3}{t-3} \cdot \frac{\sqrt{t+6}+3}{\sqrt{t+6}+3} = \lim_{t\to 3} \frac{t+6-9}{(t-3)(\sqrt{t+6}+3)} = \boxed{\frac{1}{6}}$

b) $\lim_{x\to -7} \frac{7+x}{\frac{1}{7}+\frac{1}{x}} = \lim_{x\to -7} \frac{7+x}{1} \cdot \frac{7x}{x+7} = \boxed{-49}$

c) $\lim_{t\to 0} \frac{\sqrt{t^2+25}-5}{t^2} \cdot \frac{\sqrt{t^2+25}+5}{\sqrt{t^2+25}+5} = \lim_{t\to 0} \frac{\cancel{t^2}+25-25}{\cancel{t^2}(\sqrt{t^2+25}+5)} = \boxed{\frac{1}{10}}$

d) $\lim_{t\to 0} \frac{2}{t^2+t} - \frac{2}{t} = \lim_{t\to 0} \frac{2-2(t+1)}{t(t+1)} = \lim_{t\to 0} \frac{-2\cancel{t}}{\cancel{t}(t+1)}$

$= \boxed{-2}$

14) Find the following limit $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x^2}\right)$ (6 Points)

(Hint: Use the Squeeze Theorem)

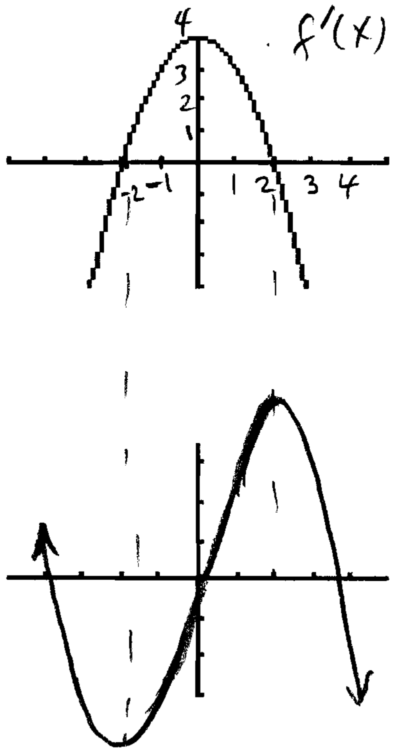
Observe that $-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

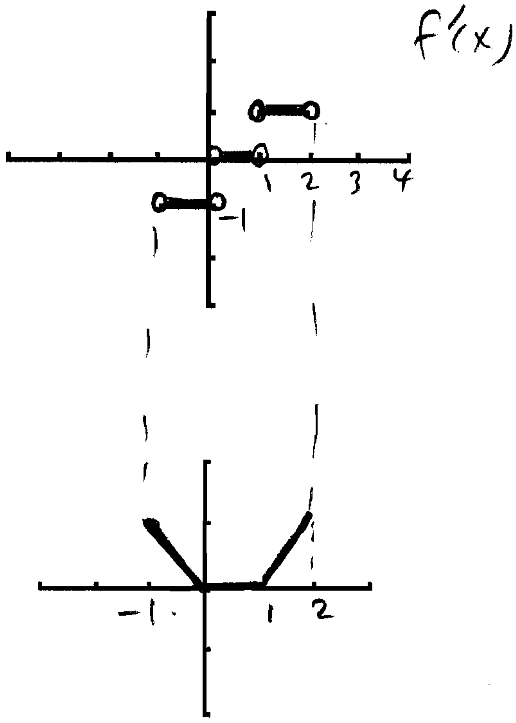
and $-x^4 \le x^4 \cos\left(\frac{1}{x^2}\right) \le x^4$

and $\lim_{x\to 0} -x^4 = 0$ and $\lim_{x\to 0} x^4 = 0$

$\therefore$ $\boxed{\text{By Squeeze theorem } \lim_{x\to 0} x^4 \cos\left(\frac{1}{x^2}\right) = 0}$

15) Given the graph of $y = f'(x)$, sketch the graph of $y = f(x)$ (12 Points)





16) Given $f(x)=\sqrt{1+3x}$ (Extra Credits 5 Points)

Find the $f'(x)$ using either of the two definitions discussed in class

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1+3(x+h)}-\sqrt{1+3x}}{h}\cdot\frac{\sqrt{1+3(x+h)}+\sqrt{1+3x}}{\sqrt{1+3(x+h)}+\sqrt{1+3x}}$$

$$= \lim_{h\to 0}\frac{\cancel{1}+3(\cancel{x}+h)-(\cancel{1}+\cancel{3x})}{h\left(\sqrt{1+3(x+h)}+\sqrt{1+3x}\right)} = \lim_{h\to 0}\frac{3\cancel{h}}{\cancel{h}\left(\sqrt{1+3(x+h)}+\sqrt{1+3x}\right)}$$

$$= \boxed{\frac{3}{2\sqrt{1+3x}}}$$

17) Given $f(x)=\dfrac{1}{\sqrt{x+1}}$ (Extra Credits 5 Points)

Find the $f'(x)$ using either of the two definitions discussed in class.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h+1}}-\dfrac{1}{\sqrt{x+1}}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\left[\left(\sqrt{x+h+1}\right)\left(\sqrt{x+1}\right)\right]}\cdot\frac{\left(\sqrt{x+1}+\sqrt{x+h+1}\right)}{\left(\sqrt{x+1}+\sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0}\frac{x+1-(x+h+1)}{h\left[\left(\sqrt{x+h+1}\right)\left(\sqrt{x+1}\right)\right]\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} = \frac{-1}{(x+1)\left(2\sqrt{x+1}\right)}$$

$$= \boxed{\frac{-1}{2(x+1)^{3/2}}}$$