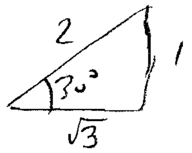


Professor Katiraie Calculus I Spring 2008 Test V Form A (chapters 1 -- 5)

Name Solutions

Total Possible Points = 140

☺☺☺ **Plus 10 Points Extra Credit** ☺☺☺

1) Find the equation of the tangent line to the curve $y = 2x\cos x$, at the point $(\pi, -2\pi)$. (10 Points)

$$y' = 2\cos x - 2x\sin x$$

$$y' = m = 2\cos\pi - 2\pi\underbrace{\sin\pi}_{0} = -2$$

$$y - -2\pi = -2(x - \pi)$$

$$y + 2\pi = -2x + 2\pi \implies \boxed{y = -2x}$$

2) A particle starts at the origin and moves along the parabola $y = x^2$ such that its distance from the origin increases at 5 units per second. How fast is its x-coordinate changing as it passes through the point (5 , 25)? (10 points)

(x, x^2)

$(0, 0)$

$$D = \sqrt{(x-0)^2 + (x^2-0)^2} \implies D^2 = x^2 + x^4$$

$$\cancel{2}D\frac{dD}{dt} = \cancel{2}x\frac{dx}{dt} + 4x^3\frac{dx}{dt}$$

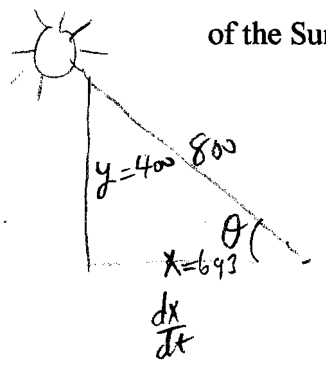
$$D = \sqrt{5^2 + 25^2}$$

$$D = \sqrt{650}$$

$$\sqrt{650}(5) = 5\frac{dx}{dt} + 2(5)^3\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{5\sqrt{650}}{255} \simeq \boxed{0.5\ \frac{\text{Units}}{\text{sec}}}$$

3) The angle of elevation of the Sun is decreasing at a rate of 0.25 rad/hour. How fast is the shadow cast by a 400–foot–tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{6}$ (10 points)



$$\frac{d\theta}{dt} = -0.25$$

$$\tan\theta = \frac{400}{x} \implies \tan\theta = 400x^{-1}$$

$$\sec^2\theta\frac{d\theta}{dt} = -400x^{-2}\frac{dx}{dt}$$

$$\tan\frac{\pi}{6} = \frac{400}{x} \implies x = \frac{400}{\tan\frac{\pi}{6}} = 693$$

$$\frac{dx}{dt} = \frac{\left(\frac{800}{693}\right)^2(+0.25)}{(+400)(693)^{-2}}$$

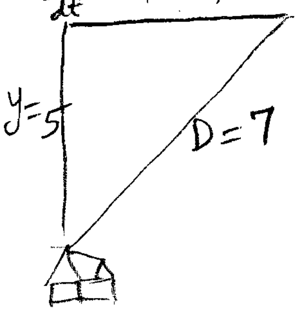
$$\boxed{\frac{dx}{dt} = 400\ \text{ft/HR}}$$

$$5^2 + x^2 = 7^2$$

$$x^2 = 24 \quad x = \sqrt{24}$$

3) A plane flying horizontally at an altitude of 5 km and a speed of 400 km/h passed directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 7 km away from the station. (10 points)

$\frac{dx}{dt} = 400\,\text{Km/HR}$

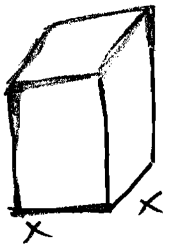


$$x^2 + y^2 = D^2$$

$$2x\frac{dx}{dt} + 0 = 2D\frac{dD}{dt}$$

$$(\sqrt{24})(400) = 7\frac{dD}{dt} \Rightarrow \frac{dD}{dt} = \frac{400\sqrt{24}}{7} = \frac{800\sqrt{6}}{7}$$

$$\frac{dD}{dt} = 279.94\ \text{Km/HR}$$



4) If 2000 sq. cm of material is available to make a box with a square base and an open top, find the largest possible volume of the box. (10 points)

$$x^2 + 4xy = 2000$$

$$y = \frac{2000 - x^2}{4x} = 500x^{-1} - 0.25x$$

$$V = x^2 y = x^2(500x^{-1} - 0.25x)$$

$$V = 500x - 0.25x^3$$

$$\frac{dV}{dx} = 500 - 0.75x^2 = 0$$

$$0.75x^2 = 500$$

$$x^2 = \frac{500}{0.75}$$

$$x \cong 25.82\ \text{cm}$$

$$y \cong \frac{2000 - 25.82^2}{4(25.82)} = 19.91\ \text{cm}$$

$$V_{max} = 8606.63\ \text{cm}^3$$

5) Find the points on the ellipse $6x^2 + y^2 = 8$ that are farthest away from the point (2,0) (10 points)

$$y^2 = -6x^2 + 8$$

$$D = \sqrt{(x-2)^2 + (y-0)^2}$$

$$D^2 = (x-2)^2 + y^2$$

$$D^2 = x^2 - 4x + 4 + -6x^2 + 8$$

$$D^2 = -5x^2 - 4x + 12$$

$$2D\frac{dD}{dx} = -10x - 4 = 0$$

$$-10x = 4$$

$$x = -0.4$$

$$y^2 = -6(-0.4)^2 + 8$$

$$y \cong \pm 2.65$$

$(-0.4, 2.65)$
$(-0.4, -2.65)$

$$f'(x) = 12\sqrt{x} - 10x^{\frac{3}{2}} = 12x^{\frac{1}{2}} - 10x^{\frac{3}{2}}$$

9) Given $f'(x) = 2\sqrt{x} \bullet (6-5x)$ and f(1) = 7; Find $f(x)$ (5 points)

$$f(x) = \frac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$f(x) = \frac{24}{3}x^{\frac{3}{2}} - 4x^{\frac{5}{2}} + c \Rightarrow \boxed{f(x) = 8x^{\frac{3}{2}} - 4x^{\frac{5}{2}} + 3}$$

$$f(x) = 8x^{\frac{3}{2}} - 4x^{\frac{5}{2}} + c$$

$$f(1) = 7 \Rightarrow 7 = 8 - 4 + c \Rightarrow C = 3$$

10) Given $f''(x) = 5x^{-2}$, $x > 0$, $f(2) = 3$, $f(4) = 0$
Find $f(x)$ (5 points)

$$f'(x) = \frac{5x^{-1}}{-1} + c \qquad f(x) = -5\ln x + 0.23x + 6.01$$

$$f(x) = -5\ln x + cx + D$$

$$f(2) = 3 \Rightarrow 3 = -5\ln 2 + 2c + D \Rightarrow 2c + D = 3 + 5\ln 2$$

$$f(4) = 0 \Rightarrow 0 = -5\ln 4 + 4c + D \overset{\ominus}{\Rightarrow} 4c + D = 5\ln 4$$

$$\begin{cases} 2c + D = 3 + 5\ln 2 \\ -4c - D = -5\ln 4 \end{cases} \qquad -2c = 3 + 5\ln 2 - 5\ln 4$$

ADD

$$c \cong 0.23 \qquad D \cong 6.01$$

11) If $\int_0^3 f(x)dx = 21$, $\int_0^6 g(x)dx = 4$, and $\int_0^3 g(x)dx = 7$

a) Find the value of $\int_3^0 f(x) * g(x)dx$ (5 points)

Impossible Based on the Given Information

b) Find the value of $\int_3^0 (f(x) - g(x))dx$ (5 points)

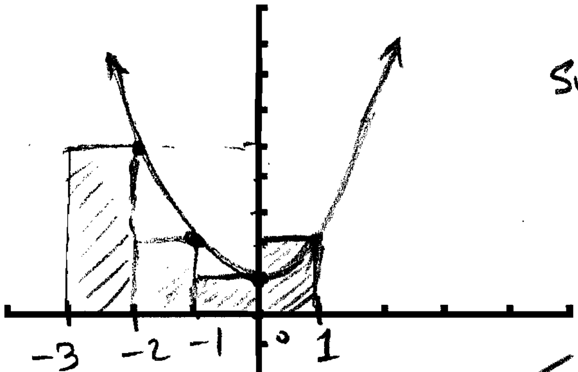
$$\int_3^0 f(x)dx - \int_3^0 g(x)dx$$

$$-21 - -7 = -21 + 7 = \boxed{-14}$$

$\Delta X = \frac{1 - -3}{4} = 1$

12) Given the function $f(x) = 1 + x^2$, $-3 \le x \le 1$
Estimate the area under the graph of f(x) using 4
(hint: n = 4)approximating rectangles and taking the sample points to be:

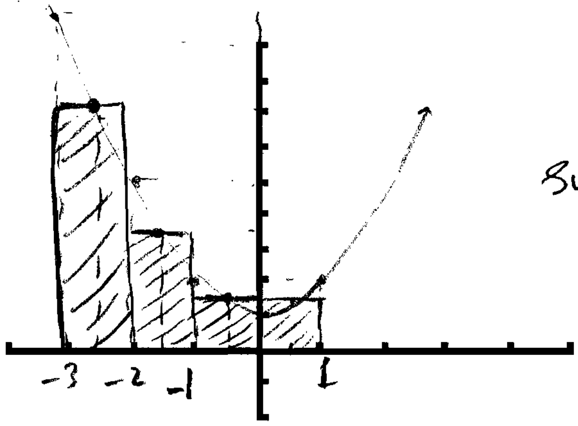
a) Right endpoints (Draw the appropriate rectangles and find the area) (5 points)



Sum seq((1+x²).1, x, -3+1, 1, 1) = 10

$1(5)+1(2)+(1)(1)+(1)(2) = 10$

b) Midpoints (Draw the appropriate rectangles and find the area) (5 points)



Sum seq((1+x²)1, x, -3+½, 1-½, 1)

$\boxed{= 13}$

13) Given $\int_a^b x\,dx = 30$ and $\int_b^a 5\,dx = -30$ Find a and b (10 points)

$a = 2$
$b = 8$

$\frac{x^2}{2}\Big|_a^b = 30$ $\qquad \int_a^b 5dx = 30 \Rightarrow 5b - 5a = 30$

$\frac{b^2}{2} - \frac{a^2}{2} = 30$ $\Rightarrow \begin{cases} b^2 - a^2 = 60 \\ b - a = 6 \end{cases} \Rightarrow b = a + 6$

$\begin{cases} b^2 - a^2 = 60 \\ 5b - 5a = 30 \end{cases}$

$(a+6)^2 - a^2 = 60$ $\qquad b = 8$

$\cancel{a^2} + 12a + 36 - \cancel{a^2} = 60$

$12a = 24$ $\quad a = 2$

$$Water = \int (200-4t)\,dt$$

14) Water flows from the bottom of a storage tank at a rate of

$r(t)=200-4t$ $\dfrac{\text{liters}}{\text{minute}}$, Where $0 \le t \le 50$ minutes (10 points)

a) Find the amount of water that flows from the tank initially (at time t = 0).

$$\int_0^0 (200-4t)\,dt = \boxed{0 \text{ liters}}$$

b) Find the amount of water that flows from the tank during the first 25 minutes.

$$\int_0^{25} (200-4t)\,dt = 200t - \frac{4t^2}{2}\Big|_0^{25}$$

$$= \left(200(25) - 2(25)^2\right) - (0) = \boxed{3750 \text{ liters}}$$

15) Find the area enclosed by the following curves: (10 points)

$y = 2x + x^2$

and

$y = 2x + 9$

$$2x + x^2 = 2x+9$$

$$x^2 = 9$$

$$x = \pm 3$$

$$\int_{-3}^{3} (2x+9) - (2x+x^2)\,dx = \int_{-3}^{3} (9-x^2)\,dx$$

$$= 9x - \frac{x^3}{3}\Big|_{-3}^{3}$$

$$= \left(9(3) - \frac{3^3}{3}\right) - \left(9(-3) - \frac{(-3)^3}{3}\right) =$$

$$18 - (-18) = \boxed{36} \text{ units}^2$$

16) Let $f(x) = \left(\int_{2x}^{10} \sqrt{t}\,dt\right) + 100$

Find the value of $f'(10000)$ (5 points)

$$f(x) = \left(-\int_{10}^{2x} \sqrt{t}\,dt\right) + 100$$

$$f'(x) = \left(-\sqrt{2x}\right)(2) + 0$$

$$f'(10000) = -2\sqrt{2(10000)} = -2(100)\sqrt{2}$$

$$= \boxed{-200\sqrt{2}} = -282.84$$

17) The velocity of a particle moving along a line is $t^2 - 3t - 4$ meters per second.

Find the acceleration of the particle when the velocity of the particle is zero. (5 points)

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0$

$t = 4$ $\quad \cancel{t = -1}$

$a = 2t - 3$

$\boxed{a(4) = 2(4) - 3 = 5 \text{ m/sec}^2}$

18) Find the value of the integral $\int_C^D \frac{3x^2 - 5}{x} dx$ (5 points)

(Assume C >0 and D > 0, and leave your answer in terms of C and D)

$$\int_C^D \left(3x - \frac{5}{x}\right) dx = \frac{3x^2}{2} - 5\ln x \Big|_C^D$$

$$= \left(\frac{3}{2}D^2 - 5\ln D\right) - \left(\frac{3}{2}C^2 - 5\ln C\right)$$

$$= \frac{3}{2}D^2 - 5\ln D - \frac{3}{2}C^2 + 5\ln C$$

19) Determine by differentiation whether the following formula is true or false
(Must Show Procedure)

$$\int \frac{du}{u^2 + a^2} = \frac{1}{2a} \ln \left| \frac{2u + a}{u - a} \right| + C$$ (5 points)

$$\left(\frac{1}{2a} \ln \left| \frac{2u+a}{u-a} \right| + c\right)'$$

$$\frac{1}{2a} \frac{1}{\left(\frac{2u+a}{u-a}\right)} \cdot \frac{2(u-a) - 1(2u+a)}{(u-a)^2} = \frac{1}{2a} \frac{u-a}{2u+a} \frac{-3a}{(u-a)^2}$$

$$= \frac{-3}{2(2u+a)(u-a)} \neq \frac{1}{u^2 + a^2}$$

So, the Equation is false

Bonus Question:

20) Let $f(x)=\frac{1}{2}\int_{2x}^{5x}\frac{u+2}{u-1}\,du$

Find the value of $f'(0)$

(5 points)

$$\frac{1}{2}\int_{2x}^{0}\frac{u+2}{u-1}\,du+\frac{1}{2}\int_{0}^{5x}\frac{u+2}{u-1}\,du$$

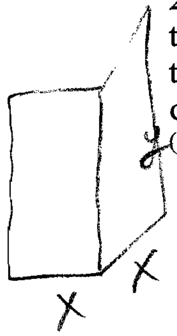
$$-\frac{1}{2}\int_{0}^{2x}\frac{u+2}{u-1}\,du+\frac{1}{2}\int_{0}^{5x}\frac{u+2}{u-1}\,du$$

$$f'(x)=-\frac{1}{2}\left(\frac{2x+2}{2x-1}\right)\cdot 2+\frac{1}{2}\left(\frac{5x+2}{5x-1}\right)\cdot 5$$

$$f'(0)=-\frac{1}{2}\left(\frac{2(0)+2}{2(0)-1}\right)\cdot 2+\frac{1}{2}\left(\frac{5(0)+2}{5(0)-1}\right)\cdot 5=2+\frac{5}{2}(-2)=\boxed{-3}$$

21) A closed box with square base is to be built to house an ant colony. The bottom and top of the box will be made of material costing \$1 per square foot, and all four sides are to be constructed of glass costing \$5 per square foot. What are the dimensions of the box of greatest volume that can be constructed for \$65?
(Round your answers to two decimal places)

(5 points)



$$\cancel{\$}1x^2+\cancel{\$}1x^2+\$5(4xy)=\cancel{\$}65$$

$$2x^2+20xy=\cancel{\$}65$$

$$y=\frac{65-2x^2}{20x}=$$

$$y=3.25x^{-1}-0.1x$$

$$V=x^2y$$

$$V=x^2(3.25x^{-1}-0.1x)$$

$$V=3.25x^{1}-0.1x^{3}$$

$$\frac{dV}{dx}=3.25-0.3x^2=0$$

$x=3.29$ feet $y\cong 0.66$ feet