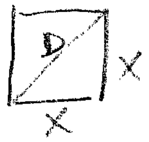
Professor Katiraie Calculus I Spring 2008 Test V Form B (chapters 1 -- 5)

Name Solution Total Possible Points = 140

☺☺☺ **Plus 10 Points Extra Credit** ☺☺☺



1) If the diagonal of a square decreases at the rate of 2 inch/second, how fast is the area changing when the side of the square is 15 inches? (10 points)

$\frac{dD}{dt} = -2 \frac{\text{inch}}{\text{sec}}$

$$x^2 + x^2 = D^2 \Rightarrow 2x^2 = D^2 \Rightarrow x^2 = \frac{D^2}{2}$$

$$A = x^2 \quad \text{But} \quad x^2 = \frac{D^2}{2}$$

$$A = \frac{D^2}{2}$$

$$\frac{dA}{dt} = \frac{1}{2} \cdot 2D \frac{dD}{dt} \Rightarrow \frac{dA}{dt} = \sqrt{15^2 + 15^2}(-2)$$

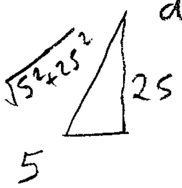
$$\boxed{= -30\sqrt{2} \ \frac{\text{in}^2}{\text{sec}}}$$

2) A particle starts at the origin and moves along the parabola $y = x^2$ such that its distance from the origin increases at 5 units per second. How fast is its x-coordinate changing as it passes through the point (5 , 25)? (10 points)

$$D = \sqrt{(x-0)^2 + (y-0)^2} \qquad \frac{dD}{dt} = 5 \ \frac{\text{units}}{\text{sec}} \qquad \frac{dx}{dt} = ?$$

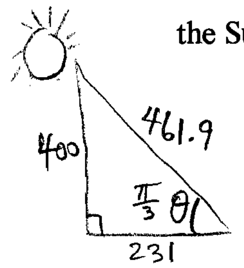
$$D^2 = x^2 + x^4$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 4x^3\frac{dx}{dt} \Rightarrow D\frac{dD}{dt} = x\frac{dx}{dt} + 2x^3\frac{dx}{dt}$$



$$\frac{dx}{dt} = \frac{D\frac{dD}{dt}}{x + 2x^3} = \frac{\sqrt{5^2+25^2}\,(5)}{5 + 2(5)^3} = \boxed{0.5 \ \frac{\text{units}}{\text{sec}}}$$

3) The angle of elevation of the Sun is decreasing at a rate of 0.2 rad/hour. How fast is the shadow cast by a 400–foot–tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{3}$ (10 points)



$\tan\frac{\pi}{3} = \frac{400}{x}$

$$\frac{d\theta}{dt} = -0.2 \qquad \frac{dx}{dt} = ?$$

$$\tan\theta = \frac{400}{x} = 400x^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = -400x^{-2}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{\sec^2\theta \frac{d\theta}{dt}}{-400x^{-2}}$$

$$\frac{dx}{dt} = \frac{(4)(-0.2)}{-400(231)^{-2}} = \boxed{106.722 \text{ feet/sec}}$$

4) A rectangular field is to be enclosed on four sides with a fence. Fencing costs $10 per foot for two opposite sites, and $5 per foot for the other two sides. Find the dimensions of the field of area 730 sq. ft. that would be the cheapest to enclose.

(10 points)

$20x + 10y = \text{Cost}$

$xy = 730$

$y = \frac{730}{x} = 730x^{-1}$

$\text{Cost} = 20x + 10(730x^{-1})$

$\text{Cost} = 20x + 7300x^{-1}$

$\text{Cost}' = 20 - 7300x^{-2} = 0$

$\frac{7300}{x^2} = 20$

$x = 19.10$ feet

$y = 38.2$ feet

19.10 feet by 38.2 feet

5)Analytically find the exact value of all critical numbers of the following functions. (In other words, find the x-coordinates of the critical points.)

(10 points)

$y = x^{\frac{4}{5}}(x-4)^2$

$y' = \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + 2x^{\frac{4}{5}}(x-4)$

$y' = \frac{4(x-4)^2}{5x^{1/5}} + 2x^{\frac{4}{5}}(x-4)$

$y' = \frac{4(x-4)^2}{5x^{\frac{1}{5}}} + \frac{10x^1(x-4)}{5x^{1/5}}$

$y' = \frac{4(x^2-8x+16)+10x^2-40x}{5x^{\frac{1}{5}}} = \frac{14x^2-72x+64}{5x^{\frac{1}{5}}}$

$x = \frac{72 \pm \sqrt{72^2-4(14)(64)}}{2(14)} = \begin{cases} 4 \\ 1.143 \end{cases}$

Critical Numbers $\Rightarrow x = 0,\ x = 4,\ x = 1.143$

$R(x) = x(25-x) = 25x - x^2$

6) A company has cost function $C(x) = 100 - 14x + x^2$ and demand function $p(x) = 25 - x$, where x is the number of staplers and $p(x)$ is in dollars.

a. How many units should the company make to maximize its profit? (5 points)

$P(x) = 25x - x^2 - (100 - 14x + x^2)$

$P(x) = 25x - x^2 - 100 + 14x - x^2$

$P(x) = -2x^2 + 39x - 100 \qquad P'(x) = -4x + 39 = 0$

$x = 9.75$ units ≈ 10 units

b. How much is the maximum profit? (3 points)

$\boxed{P(9.75) = \$90.125}$ OR $\boxed{P(10) = \$90}$

c. What price would produce maximum profit? (3 points)

$\rightarrow$ Price $= 25 - x = 25 - 9.75 = \boxed{\$15.25}$

OR

Price $= 25 - x = 25 - 10 = \boxed{\$15}$

7) Given $f'(x) = 2\sqrt{x} \bullet (6-5x)$ and $f(1) = 7$; Find $f(x)$ (5 points)

$f'(x) = 12\sqrt{x} - 10x^{\frac{3}{2}}$

$f(x) = \dfrac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{10x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$f(x) = 8x^{\frac{3}{2}} - 4x^{\frac{5}{2}} + c$

$f(1) = 7 \qquad \Rightarrow 7 = 8 - 4 + c \quad \Rightarrow C = 3$

$\boxed{f(x) = 8x^{\frac{3}{2}} - 4x^{\frac{5}{2}} + 3}$

8) Given $f''(x) = 5x^{-2}$, $x > 0$, $f(2) = 3$, $f(4) = 0$
Find $f(x)$ (5 points)

$f'(x) = \dfrac{5x^{-1}}{-1} + c$

$f(x) = -5\ln x + cx + D$

$f(2) = 3 \Rightarrow 3 = -5\ln 2 + 2c + D \quad \Rightarrow \begin{cases} 2c + D = 3 + 5\ln 2 \\ 4c + D = 5\ln 4 \end{cases} \Rightarrow \begin{cases} 2c + D = 3 + 5\ln 2 \\ -4c - D = -5\ln 4 \end{cases}$

$f(4) = 0 \Rightarrow 0 = -5\ln 4 + 4c + D$

ADD these

$-2c = 3 + 5\ln 2 - 5\ln 4$

$c \approx 0.23$ and $\boxed{D \cong 6.01}$

9) If $\int_0^3 f(x)dx = 21$, $\int_0^6 g(x)dx = 4$, and $\int_0^3 g(x)dx = 7$; $\int_3^0 f(x)\,dx = -21$

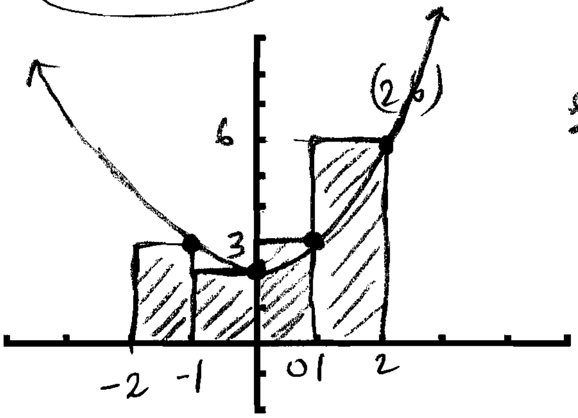
a) Find the value of $\int_3^0 f(x) * g(x)dx$ (5 points)

Impossible Based on the given Information

b) Find the value of $\int_3^0 (f(x) - g(x))dx$ $= -21 - -7 = -21 + 7 = \boxed{-14}$ (5 points)

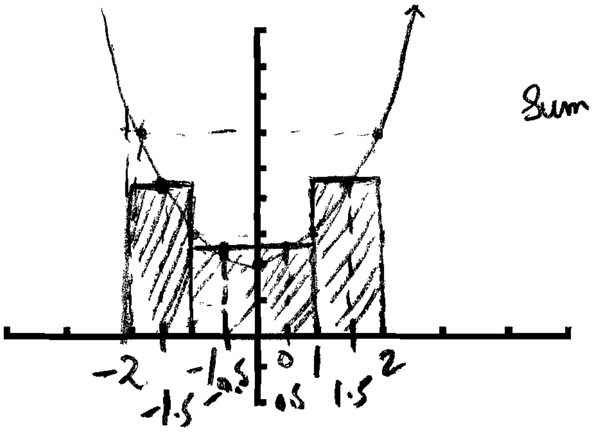
10) Given the function $f(x) = 2 + x^2$, $-2 \le x \le 2$
Estimate the area under the graph of f(x) using 4
(hint: n = 4)approximating rectangles and taking the sample points to be:

a) Right endpoints (Draw the appropriate rectangles and find the area) (5 points)



Sum Seq $((2+x^2)*1, x, -2+1, 2, 1)$
$\boxed{= 14 \text{ units}^2}$

b) Midpoints (Draw the appropriate rectangles and find the area) (5 points)



Sum Seq $((2+x^2)*1, x, -2+\frac{1}{2}, 2-\frac{1}{2}, 1)$
$\boxed{= 13 \text{ units}^2}$

11) Given $\int_a^b x\,dx = 36$ and $\int_b^a 5\,dx = -15$ Find a and b. (10 points)

$\frac{x^2}{2}\Big|_a^b = 36 \Rightarrow \frac{b^2}{2} - \frac{a^2}{2} = 36 \Rightarrow b^2 - a^2 = 72$

$5x\Big|_b^a = -15 \Rightarrow 5a - 5b = -15 \Rightarrow a - b = -3$

$\boxed{a = b - 3}$

$b^2 - (b-3)^2 = 72 \Rightarrow \cancel{b^2} - \cancel{b^2} + 6b - 9 = 72$ $\boxed{b = 13.5}$

$6b = 81$ $\boxed{a = 10.5}$

12) Water flows from the bottom of a storage tank at a rate of

$r(t) = 200 - 4t \ \frac{\text{liters}}{\text{minute}}$, Where $0 \le t \le 50$ minutes (10 points)

a) Find the amount of water that flows from the tank initially (at time t = 0).

$\int_0^0 (200 - 4t)\,dt = 0$ liters

b) Find the amount of water that flows from the tank during the first 25 minutes.

$\int_0^{25} (200 - 4t)\,dt = 200t - \frac{4t^2}{2} = 200t - 2t^2 \Big|_0^{25}$

$= 200(25) - 2(25)^2 - 0 = 3750$ liters

13) Find the area enclosed by the following curves: (10 points)

$y = 2x + x^2$

and

$y = 2x + 9$

$2x + x^2 = 2x + 9$

$x^2 = 9$

$x = \pm 3$

$\int_{-3}^{3} (2x+9) - (2x + x^2)\,dx = \int_{-3}^{3} (9 - x^2)\,dx = 9x - \frac{x^3}{3}\Big|_{-3}^{3}$

$= \left(9(3) - \frac{3^3}{3}\right) - \left(9(-3) - \frac{(-3)^3}{-3}\right) = 18 - -18 = 36 \text{ units}^2$

14) Let $f(x) = \left(\int_{2x}^{10} \sqrt{t}dt \right) + 100$

Find the value of $f'(10000)$ (5 points)

$f(x) = \left(-\int_{10}^{2x} \sqrt{t}\, dt \right) + 100$

$f'(x) = -\sqrt{2x} \cdot 2 + 0$

$f'(10000) = -2\sqrt{20000} + 0 = -200\sqrt{2} = \boxed{-282.84}$

15) The velocity of a particle moving along a line is $t^2 - 3t - 4$ meters per second.

Find the velocity of the particle when the acceleration of the particle is zero. (5 points)

$v = t^2 - 3t - 4$; $a = 2t - 3$

$2t - 3 = 0$ $t = \frac{3}{2} = 1.5$ second

$v = (1.5)^2 - 3(1.5) - 4 = \boxed{-6.25\ m/sec}$

16) Find the value of the integral $\int_C^D \frac{3x^2 - 5}{x} dx$ (5 points)

(Assume C >0 and D > 0, and leave your answer in terms of C and D)

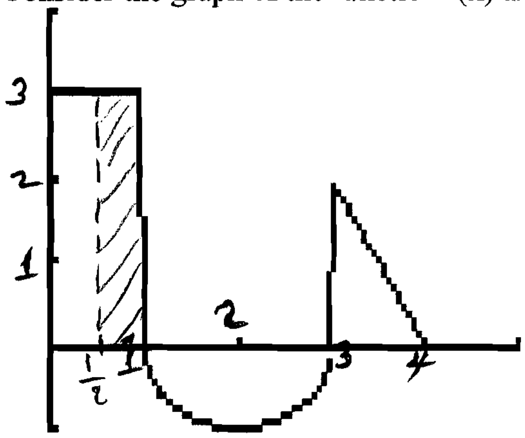
$\int_C^D \left(\frac{3x^2 - 5}{x} \right) dx = \int_C^D \left(\frac{3x^2}{x} - \frac{5}{x} \right) dx$

$= \int_C^D \left(3x - 5\left(\frac{1}{x}\right) \right) dx$

$= \frac{3x^2}{2} - 5\ln x \Big|_C^D = \left(\frac{3}{2}D^2 - 5\ln D \right) - \left(\frac{3}{2}C^2 - 5\ln C \right)$

$= \boxed{\frac{3}{2}D^2 - 5\ln D - \frac{3}{2}C^2 + 5\ln C}$

17) Consider the graph of the function f(x) and



Using geometry compute the following: (5 Points)

a) $\int_{1/2}^{3} f(x)\,dx = \frac{1}{2}(3) - \frac{\pi(1)^2}{2} = \boxed{\frac{3}{2} - \frac{\pi}{2}}$

b) $\int_{0}^{2} f(x)\,dx = 1(3) - \frac{1}{4}\pi(1)^2 = \boxed{3 - \frac{\pi}{4}}$

c) $\int_{1}^{4} f(x)\,dx = -\frac{1}{2}(\pi)(1)^2 + \frac{1}{2}(1)(2)$

$= \boxed{-\frac{\pi}{2} + 1}$

d) $\int_{4}^{0} f(x)\,dx = \boxed{-4 + \frac{\pi}{2}}$

$\int_0^4 f(x)\,dx = (1)(3) - \frac{1}{2}\pi(1)^2 + \frac{1}{2}(1)(2)$

$\int_0^4 f(x)\,dx = 4 - \frac{\pi}{2}$

e) $\int_{4}^{4} f(x)\,dx = \boxed{0}$

Bonus Question:

18) Let $$f(x)=\frac{1}{2}\int_{2x}^{5x}\frac{u+2}{u-1}\,du$$

Find the value of $f'(0)$

(5 points)

$$f(x)=\frac{1}{2}\int_{2x}^{0}\frac{u+2}{u-1}\,du+\frac{1}{2}\int_{0}^{5x}\frac{u+2}{u-1}\,du$$

$$=-\frac{1}{2}\int_{0}^{2x}\frac{u+2}{u-1}\,du+\frac{1}{2}\int_{0}^{5x}\frac{u+2}{u-1}\,du$$

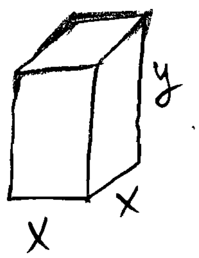
$$f'(x)=-\frac{1}{\cancel{2}}\left(\frac{2x+2}{2x-1}\right)\cancel{2}+\frac{1}{2}\,\frac{5x+2}{5x-1}\cdot 5$$

$$f'(0)=2+-5=-3$$

Bonus Question:

19) A closed box with square base is to be built to house an ant colony. The bottom and top of the box will be made of material costing \$1 per square foot, and all four sides are to be constructed of glass costing \$5 per square foot. What are the dimensions of the box of greatest volume that can be constructed for \$65?
(Round your answers to two decimal places)

(5 points)



$$\$1x^2+\$1x^2+\$5(4xy)=65$$

$$2x^2+20xy=65 \Rightarrow y=\frac{65-2x^2}{20x}$$

$$y=3.25x^{-1}-0.1x$$

$$V=x^2y$$

$$V=x^2(3.25x^{-1}-0.1x)=3.25x-0.1x^3$$

$$\frac{dV}{dx}=3.25-0.3x^2=0 \qquad x^2=\frac{3.25}{0.3}$$

$$x\cong 3.29\text{ feet}$$

$$y\cong 0.66\text{ feet}$$