Calculus I Test IV Professor: Fred Katiraie Date: November 16th/ 2005

Name: Key

Total Possible Points = 140

This is not a graphing calculator test. I will not give credit to answers not supported by your work, i.e. No Procedure = No Points

1) Find the derivative of the following functions: **(4 Points each)**

(Do Not Simplify)

a. $y = 10^{\sec \pi\theta}$

$y' = 10^{\sec \pi\theta}(\ln 10)\sec(\pi\theta)\tan(\pi\theta)\cdot\pi$

b. $y = \cos^5(4x)$

$y' = 5\cos^4(4x)(-\sin(4x))(4)$

$= -20\cos^4(4x)\sin(4x)$

c. $y = \sin(\cot\sqrt{1+x^2})$

$y' = \cos\left[\cot(1+x^2)^{1/2}\right]\left(-\csc^2(1+x^2)^{1/2}\right)\left(\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\right)(2x)$

$= \cos\left(\cot(1+x^2)^{1/2}\right)\dfrac{-x\csc^2(1+x^2)^{1/2}}{(1+x^2)^{1/2}}$

d. $y = \sqrt{x+\sqrt{x}} = (x+x^{1/2})^{\frac{1}{2}}$

$y' = \frac{1}{2}(x+x^{\frac{1}{2}})^{-\frac{1}{2}}(1+\frac{1}{2}x^{-\frac{1}{2}})$

$= \dfrac{\left(1+\frac{1}{2\sqrt{x}}\right)}{\left(2(x+x^{1/2})^{1/2}\right)}$

2a) Find the linearization of $f(x) = \sqrt[3]{1+3x}$ at $a = 0$. (4 points)
State the corresponding linear approximation.

$f'(x) = \frac{1}{\cancel{3}}(1+3x)^{-2/3}(\cancel{3}) \Rightarrow f'(0) = 1 \; ; \; f(0) = 1$

$L = f(0) + f'(0)(x-0)$

$= 1 + 1(x-0)$

$L = 1 + x \Longrightarrow \boxed{\sqrt[3]{1+3x} \approx 1+x}$

2b) Use the above to give an approximate value for $\sqrt[3]{1.03}$. (4 Points)

$\sqrt[3]{1.03} = \sqrt[3]{1+3x}$

$1.03 = 1+3x$

$0.03 = 3x \Longrightarrow \boxed{\sqrt[3]{1.03} \approx 1 + 0.01 = 1.01}$

$0.01 = x$

3) At what point on the curve $y = [\ln(x+4)]^2$ is the tangent horizontal? (10 Points)

$\boxed{(-3, 0)}$

$y' = 2\ln(x+4)\frac{1}{x+4} = \frac{2\ln(x+4)}{x+4} = 0$

$2\ln(x+4) = 0 \Longrightarrow \ln(x+4) = 0 \Longrightarrow e^0 = x+4$

$x = 1 - 4 = -3$

$y = [\ln(1)]^2 = 0$

4) Show that the following curves are orthogonal. (6 points)

$2x^2 + y^2 = 3 \Longrightarrow 4x + 2yy' = 0 \Longrightarrow y' = \frac{-4x}{2y} = \boxed{-\frac{2x}{y}}$

$x = y^2 \longrightarrow 1 = 2yy' \Longrightarrow \boxed{y' = \frac{1}{2y}}$

Now $2x^2 + y^2 = 3$

$2x^2 + x = 3 \Rightarrow 2x^2 + x - 3 = 0$

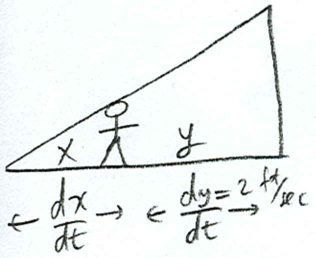
$(2x+3)(x-1) = 0 \rightarrow 1 = y^2 \rightarrow (1,1)$ & $(1,-1)$

$\cancel{x = -\frac{3}{2}}$ $x = 1$ $\qquad \boxed{\pm 1 = y}$

$\begin{cases} m_1 = \frac{1}{2} \\ m_2 = -2 \end{cases} \qquad \begin{cases} m_1 = \frac{1}{-2} \\ m_2 = 2 \end{cases}$

For Problems 5 - 7: Pick two of the following three problems.
Please, Clearly indicate the two problems you want to be graded.

5) A man 6 ft tall walks at a rate of 2 ft/sec away from a lamppost that is 23 feet high. At what rate is the length of his shadow changing when he is 70 feet away from the lamppost. (10 points)



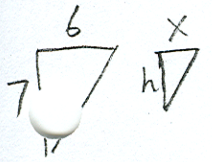
$$\frac{6}{x} = \frac{23}{x+y} \Rightarrow 6x + 6y = 23x$$

$$6y = 17x \Rightarrow y = \frac{17}{6}x$$

$$\frac{dy}{dt} = \frac{17}{6}\frac{dx}{dt}$$

$$2\,\frac{ft}{sec} = \frac{17}{6}\frac{dx}{dt} \Rightarrow \boxed{\frac{dx}{dt} = \frac{12}{17}\ ft/sec}$$

6) A container in the shape of an inverted right circular cone has a radius of 6.00 inches at the top and a height of 7.00 inches. At the instant when the water in the container is 5.00 inches deep, the surface level is falling at the rate of -0.700 in./sec. Find the rate at which water is being drained. (10 points)

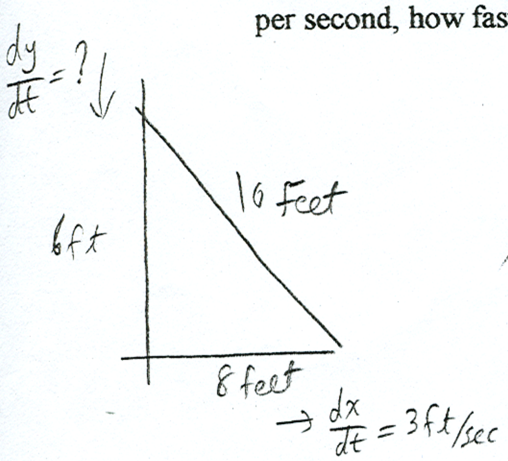


$$\frac{6}{7} = \frac{x}{h} \Rightarrow x = \frac{6}{7}h$$

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{6}{7}h\right)^2 h = \frac{36}{147}\pi h^3$$

$$\frac{dV}{dt} = \frac{36\pi}{147}3h^2\frac{dh}{dt} = \frac{36\pi}{147}(3)(5)^2(-0.7) = \boxed{-40.39\ \frac{inch^3}{sec}}$$

7) A ladder 10 feet long is leaning against a wall, with the foot of the ladder 8 feet away from the wall. If the foot of the ladder is being pulled away from the wall at 3 feet per second, how fast is the top of the ladder sliding down the wall? (10 points)



$$10^2 = 8^2 + y^2 \Rightarrow \boxed{y = 6\ feet}$$

$$c^2 = a^2 + b^2$$

$$\cancel{2}c\frac{dc}{dt} = \cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt}$$

$$0 = (8\ ft)\left(\frac{3\ ft}{sec}\right) + (6\ ft)\left(\frac{dy}{dt}\right)$$

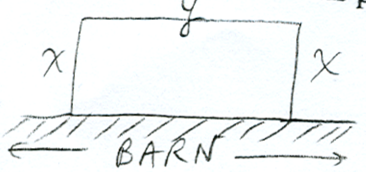
$$0 = 24\,\frac{ft^2}{sec} + 6\ ft\,\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{-24\,\frac{ft^{\cancel{2}}}{sec}}{6\ \cancel{ft}}$$

$$\boxed{\frac{dy}{dt} = -4\ ft/sec}$$

∴ Hence, the top of the ladder is sliding down the wall at a rate of $\boxed{4\ ft/sec}$

For problems 8 - 10: Pick two of the following three problems. Please, Clearly indicate the two problems you want to be graded.

8) A farmer has 20 feet of fence, and he wishes to make from it a rectangular pen for his pig Wilbur, using a barn as one of the sides. In square feet, what is the maximum area possible for this pen?



(10 points)

$2x + y = 20 \text{ feet} \Longrightarrow y = 20 - 2x$

$A = xy = x(20-2x) = 20x - 2x^2$

$\frac{dA}{dx} = 20 - 4x = 0 \Longrightarrow x = \frac{-20}{-4} = 5 \text{ feet}$

But $y = 20 - 2x = 20 - 2(5) = 10$ feet

So, the Max Area is $(5 \text{ feet})(10 \text{ feet}) = \boxed{50 \text{ ft}^2}$

9) A square is to be cut from each corner of a piece of paper which is 8 cm by 10 cm, and the sides are to be folded up to create an open box. What should the side of the square be for maximum volume? (State your answer correct to two decimal places.

(10 points)

$V = x(10-2x)(8-2x) = (10x - 2x^2)(8-2x) = 80x - 20x^2 - 16x^2 + 4x^3$

$V = 4x^3 - 36x^2 + 80x$

$\frac{dV}{dx} = 12x^2 - 72x + 80 = 0$

$3x^2 - 18x + 20 = 0$

$\boxed{x = 1.47 \text{ cm}}$

$\cancel{x = 4.53 \text{ cm}}$

10) Find the point on the line $y = 2x - 3$ that is nearest to the origin?

(10 points)

$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x-0)^2 + (2x-3)^2}$

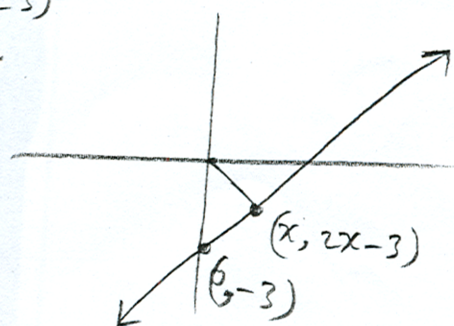
$D = \sqrt{x^2 + 4x^2 - 12x + 9} = \sqrt{5x^2 - 12x + 9}$

$\frac{dD}{dx} = \frac{1}{2}(5x^2 - 12x + 9)^{-\frac{1}{2}}(10x - 12) = 0$

$10x - 12 = 0 \Longrightarrow x = \frac{12}{10} = \boxed{1.2}$

and $y = 2(x) - 3 = 2(1.2) - 3 = 2.4 - 3 = \boxed{-0.6}$

So, Coordinate of Point is $\boxed{(1.2, -0.6)}$



11) The acceleration of an object dropped on planet Happiness is -5.00 ft/sec^2
An object is thrown upwards at 40 ft/sec from a height of 4 ft. above the surface of the planet.

(10 Points, i.e. 2.5 Points Each):

$a(t) = -5.00 \frac{ft}{sec^2}$ $V(0) = 40$ ft/sec $\boxed{S(0) = 4 \text{ feet}}$

a. Find an equation for its velocity.

$V(t) = -5t + C$ But $V(0) = 40 \Rightarrow \boxed{V(t) = -5t + 40 \text{ ft/sec}}$

b. Find an equation for its position.

$S(t) = -\frac{5t^2}{2} + 40t + C$ But $S(0) = 4 \Rightarrow \boxed{S(t) = -2.5t^2 + 40t + 4 \text{ ft}}$

c. How high does the object go?

$t = \frac{-b}{2a} = \frac{-40}{2(-2.5)} = 8$ sec; $\boxed{S(8) = 164 \text{ feet}}$

d. How long will it take before it hits the surface?

Let $S(t) = 0 = -2.5t^2 + 40t + 4 \Rightarrow t = \frac{-40 \pm \sqrt{40^2 - 4(-2.5)(4)}}{2(-2.5)}$

$\boxed{t = 16 \text{ seconds Later}}$

12) Use Newton's method to approximate a solution to the following equation
$x^3 + 2x = 3.1$ (You Must Show Proper Procedure for Credit) (7 Points)

$f(x) \Rightarrow Y_1 = x^3 + 2x - 3.1$

$f'(x) \Rightarrow Y_2 = 3x^2 + 2$

Guess $1 \rightarrow x$

$x = 1.02$

$x = 1.019764118$

1.019764085

$\boxed{x = 1.019764085}$

13) Given that the function $f(x) = x^3 + ax^2 + bx$ has critical numbers at $x = 2$, and $x = -1$, find a and b.

(10 points)

$3x^2 + 2ax + b = 0$

$3(-1)^2 + 2a(-1) + b = 0 \Rightarrow \begin{cases} 3 - 2a + b = 0 \\ -1(12 + 4a + b = 0) \end{cases}$

$3(2)^2 + 2a(2) + b = 0 \Rightarrow$

$-9 - 6a = 0 \Rightarrow \boxed{a = -1.5}$

$3 - 2a + b = 0$

$3 - 2(-1.5) + b = 0$

$3 + 3 + b = 0 \Rightarrow \boxed{b = -6}$

14) Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x)=(x-2)^3$ on the interval $[0,2]$.

(10 points)

$f'(x) = 3(x-2)^2$ $\quad f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{0-(-2)^3}{2} = \frac{8}{2} = 4$

$4 = 3(c-2)^2 \Longrightarrow \frac{4}{3} = (c-2)^2$

$c-2 = \pm\sqrt{\frac{4}{3}} = \pm\frac{2\sqrt{3}}{3}$

Plus Not in Domain $\boxed{c = 2 \pm \frac{2\sqrt{3}}{3}} \Rightarrow 2-\frac{2\sqrt{3}}{3}$

15) A company has cost function $C(x) = 200 - 50x + x^2$ and demand function $p(x) = 50 - x$, where x is the number of calculators and $p(x)$ is in dollars.

a) Find the revenue function. (2 points)

$R(x) = 50x - x^2$

b) Find the Profit function. (3 points)

$P(x) = (50x - x^2) - (200 - 50x + x^2) = -2x^2 + 100x - 200$

c) How many units should the company make to maximize its profit? (3 points)

$\frac{dP}{dx} = -4x + 100 = 0 \Longrightarrow 4x = 100$

$x = 25$ units

d) How much is the maximum profit? (2 points)

$P(25) = -2(25)^2 + 100(25) - 200$

$P(25) = \$1050$

So, Max profit is \$1050 and

it occurs at $x = 25$ units

16) Find the minimum value of the function $f(x) = x \ln x$ (8 points)

(Must Justify Your Answer)

$$f'(x) = 1 \ln x + x \frac{1}{x}$$

$$f(e^{-1}) = e^{-1} \ln(e^{-1}) = \frac{-1}{e}$$

$$\ln x + 1 = 0$$

$f'(x)$ — | $f'(x)$ +

$x = \frac{1}{e}$

$$\ln x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

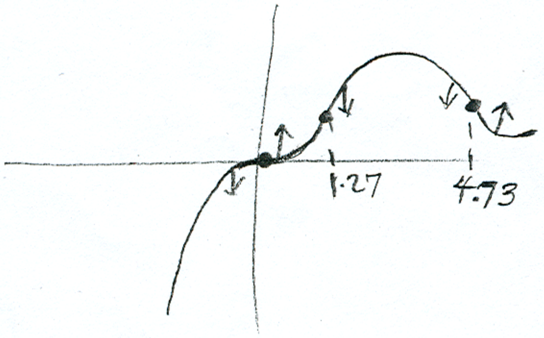
Hence, $\left(\frac{1}{e}, \frac{-1}{e}\right)$ is a minimum i.e the minimum value of $f(x)$ is $\boxed{\frac{-1}{e}}$ at the point $x = \frac{1}{e}$

17) Find all the points of inflection of $f(x) = x^3 e^{-x}$ (5 points)

(Must Justify Your Answer)

$$f'(x) = 3x^2 e^{-x} + x^3(-e^{-x})$$

$$= 3x^2 e^{-x} - x^3 e^{-x}$$



$$f''(x) = 6x e^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x} = 0$$

$$e^{-x}\left[6x - 3x^2 - 3x^2 + x^3\right] = 0$$

$$x^3 - 6x^2 + 6x = 0$$

$$x(x^2 - 6x + 6) = 0$$

$$x = \frac{6 \pm \sqrt{36 - 4(6)}}{2} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

$f''(-1) < 0$	$(0,0)$	$f''(1) > 0$
$f''(3.2) < 0$	$(3+\sqrt{3}, 0.933)$	$f''(7) > 0$
$f''(1.2) > 0$	$(3-\sqrt{3}, 0.574)$	$f''(1.3) < 0$