Calculus I Test IV Professor: Fred Katiraie Date: Spring 2006

Name: Key

This is not a graphing calculator test. I will not give credit to answers not supported by your work.

1) Given $h(x)=\sqrt{1-x}$ (5 points each)

a) Find a linearization of $h(x)=\sqrt{1-x}$ at $a=0$

$L=f(a)+f'(a)(x-a)$

$h'(x)=\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$ $\boxed{h'(0)=\frac{-1}{2}}$

$L=h(0)+h'(0)(x-0)$

$L=1+-\frac{1}{2}(x)$

$\boxed{L=1-\frac{1}{2}x}$

b) Use your answer to estimate $\sqrt{0.99}$

$\sqrt{0.99}=\sqrt{1-x}$

$.99=1-x$

$x=0.01$

$\Longrightarrow \sqrt{0.99}=1-\frac{1}{2}(0.01)=1-0.005=\boxed{0.995}$

(10 Points) 2) Find the equation of the tangent line to the curve $\sqrt{x}-\sqrt{y}=-1$ at the point (9,16).

$x^{\frac{1}{2}}-y^{\frac{1}{2}}=-1$

$\frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dx}-\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0$

$\cancel{\frac{1}{2}}\frac{1}{\sqrt{9}}\cancel{\frac{dx}{dx}}^{1}-\cancel{\frac{1}{2}}\frac{1}{\sqrt{16}}\frac{dy}{dx}=0 \Longrightarrow \frac{dy}{dx}=\frac{\frac{1}{\sqrt{9}}}{\frac{1}{\sqrt{16}}}=\frac{4}{3}$

$y-16=\frac{4}{3}(x-9)$

$y=\frac{4}{3}x-12+16$

$\boxed{y=\frac{4}{3}x+4}$

(10 points) 3) Let $y = f(x)$ be implicitly defined as $x^{\sin y} = y^{\cos x}$

Compute y' in terms of x, and y. (Hint: Use Natural Logarithms)

$$\ln x^{\sin y} = \ln y^{\cos x} \Longrightarrow \sin y \ln x = \cos x \ln y$$

$$\cos y\, y' \ln x + \sin y \frac{1}{x} = -\sin x \ln y + \cos x \frac{1}{y} y'$$

$$y'\left[\cos y \ln x - \frac{\cos x}{y}\right] = -\sin x \ln y - \frac{\sin y}{x}$$

$$\boxed{y' = \frac{-\sin x \ln y - \frac{\sin y}{x}}{\cos y \ln x - \frac{\cos x}{y}}}$$

4) Consider the circle $x^4 + y^4 = 1$. $4x^3 + 4y^3 y' = 0 \Longrightarrow \boxed{y' = \frac{-x^3}{y^3}}$

a) At what point(s) is the slope of the tangent line equal to 1? (5 points)

$$y' = 1 \Longrightarrow \frac{-x^3}{y^3} = 1 \Longrightarrow y^3 = -x^3 \Longrightarrow y = -x$$

$$(x)^4 + (-x)^4 = 1 \Longrightarrow 2x^4 = 1 \qquad x^4 = \frac{1}{2} \qquad \boxed{x = \pm\sqrt[4]{\frac{1}{2}},\ y = \mp\sqrt[4]{\frac{1}{2}}}$$

b) At what point(s) is the slope of the tangent line equal to 0? (5 points)

$$y' = 0 \Longrightarrow \frac{-x^3}{y^3} = 0 \Longrightarrow \boxed{x = 0}$$

$$x^4 + y^4 = 1$$

$$0 + y^4 = 1$$

$$y^4 = 1$$

$$\boxed{y = \pm 1} \qquad \boxed{(0, \pm 1)}$$

For Problems 5 - 7: Pick two of the following three problems. Clearly indicate the two problems you want to be graded.

5) A particle starts at the origin and moves along the parabola $y = x^2$ such that its distance from the origin increases at 4 units per second. How fast is its x-coordinate changing as it passes through the point (1,1)? (10 points)

$$D = \sqrt{(x-0)^2 + (x^2-0)^2} = \sqrt{x^2+x^4} = (x^2+x^4)^{1/2}$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2+x^4)^{-\frac{1}{2}}(2x+4x^3)\frac{dx}{dt}$$

$$4 = \frac{1}{2}(x^2+x^4)^{-\frac{1}{2}}(2x+4x^3)\frac{dx}{dt} \Longrightarrow \frac{dx}{dt} = \frac{4}{0.5(1+1)^{-1/2}(2+4)} = \frac{4\sqrt{2}}{3}\ \frac{\text{unit}}{\text{sec}}$$

6) A particle moves along a path described by $y = 4 - x^2$. At what point along the curve are x and y changing at the same rate? (10 points)

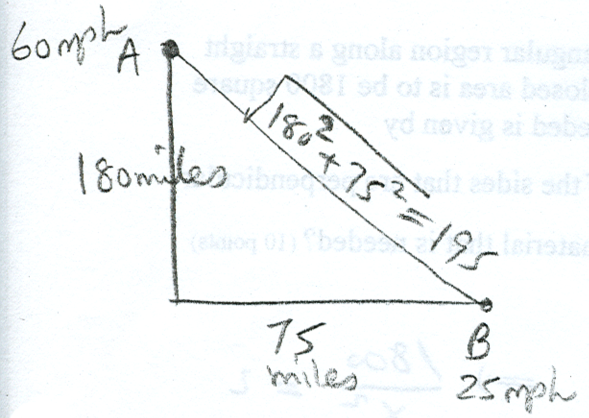
$$\frac{dy}{dt} = -2x\frac{dx}{dt} \quad \text{But} \quad \frac{dy}{dt} = \frac{dx}{dt} \Longrightarrow \frac{dx}{dt} = -2x\frac{dx}{dt} \Longrightarrow \frac{dx}{dt} = -2x\frac{dx}{dt}$$

$$\frac{dx}{dt}(1+2x) = 0 \quad \boxed{x = -\frac{1}{2}}$$

$$y = 4 - \left(\frac{-1}{2}\right)^2 = 3\frac{3}{4} = \frac{15}{4}$$

Answer: $\left(-\frac{1}{2}, \frac{15}{4}\right)$

7) Two cars leave a point at the same time. Car A goes north at 60 mph and Car B goes east at 25 mph. After 3 hours, how fast is the distance between the cars changing? (10 points)



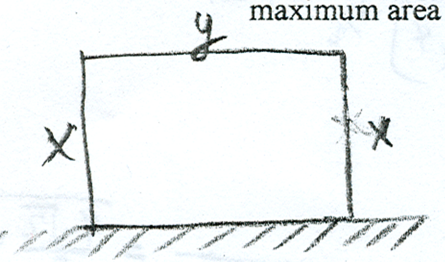
$$c^2 = a^2 + b^2$$

$$\cancel{2}c\frac{dc}{dt} = \cancel{2}a\frac{da}{dt} + \cancel{2}b\frac{db}{dt}$$

$$\frac{dc}{dt} = \frac{180(60) + (75)(25)}{195} = 65\text{mph}$$

For problems 8 - 10: Pick two of the following three problems. Clearly indicate the two problems you want to be graded.

8) A farmer has 20 feet of fence, and he wishes to make from it a rectangular pen for his pig Wilbur, using a barn as one of the sides. In square feet, what is the maximum area possible for this pen? (10 points)



$y + 2x = 20 \implies y = 20 - 2x$

$A = x(20 - 2x) = 20x - 2x^2$

$\frac{dA}{dx} = 20 - 4x \implies$ But $\frac{dA}{dx} = 0 \implies 20 - 4x = 0$

$\boxed{x = 5 \text{ feet}}$

$y = 20 - 2(5) = 10$ feet

$\boxed{\text{Max Area} = 10 * 5 = 50 \text{ ft}^2}$

9) A square is to be cut from each corner of a piece of paper which is 10 cm by 15 cm, and the sides are to be folded up to create an open box. What should the side of the square be for maximum volume? (State your answer correct to two decimal places.) (10 points)

$V = x(10 - 2x)(15 - 2x) = (10x - 2x^2)(15 - 2x)$

$V = 150x - 20x^2 - 30x^2 + 4x^3 \implies V(x) = 4x^3 - 50x^2 + 150x$

$V'(x) = 12x^2 - 100x + 150 \implies \boxed{x = 1.96 \text{ cm}}$ OR $\cancel{x = 6.37 \text{cm}}$

Impossible Case

10) A rancher intends to fence off three sides of a rectangular region along a straight stretch of river (no fence along the river). The enclosed area is to be 1800 square meters. The total length of fencing material, F, needed is given by $F = 2x + \frac{1800}{x}$, where x represents the lengths of the sides that are perpendicular to the river. What is the least amount of fencing material that is needed? (10 points)

$F = 2x + 1800x^{-1}$

$F' = 2 - 1800x^{-2} \implies 2 - \frac{1800}{x^2} = 0 \implies \frac{1800}{x^2} = 2$

$2x^2 = 1800 \implies x^2 = 900$

$\boxed{x = 30 \text{ meters}}$

Answer: $\boxed{F = 2(30) + \frac{1800}{30} = 120 \text{ meters}}$

11) Analytically find the exact value of all critical numbers of the following functions. (In other words, find the x-coordinates of the critical points.) (10 points)

a) $y = x^3 + \frac{48}{x} + 3$ $= x^3 + 48x^{-1} + 3$

$y' = 3x^2 - 48x^{-2} = 0$

$3x^2 - \frac{48}{x^2} = 0 \Rightarrow \frac{3x^4 - 48}{x^2} = 0$

$3x^4 - 48 = 0$

$3(x^4 - 16) = 0 \Rightarrow 3(x^2-4)(x^2+4) = 0$

$3(x+2)(x-2)(x^2+4) = 0$

C.P. $\boxed{x = -2}$, $\boxed{x = 2}$

b) $y = x^{2/3} - x + 3$

$y' = \frac{2}{3}x^{-\frac{1}{3}} - 1$

$= \frac{2}{3x^{\frac{1}{3}}} - 1 = \frac{2 - 3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}} = 0$

$2 - 3x^{\frac{1}{3}} = 0$

$3x^{\frac{1}{3}} = 2$

$x^{\frac{1}{3}} = \frac{2}{3}$

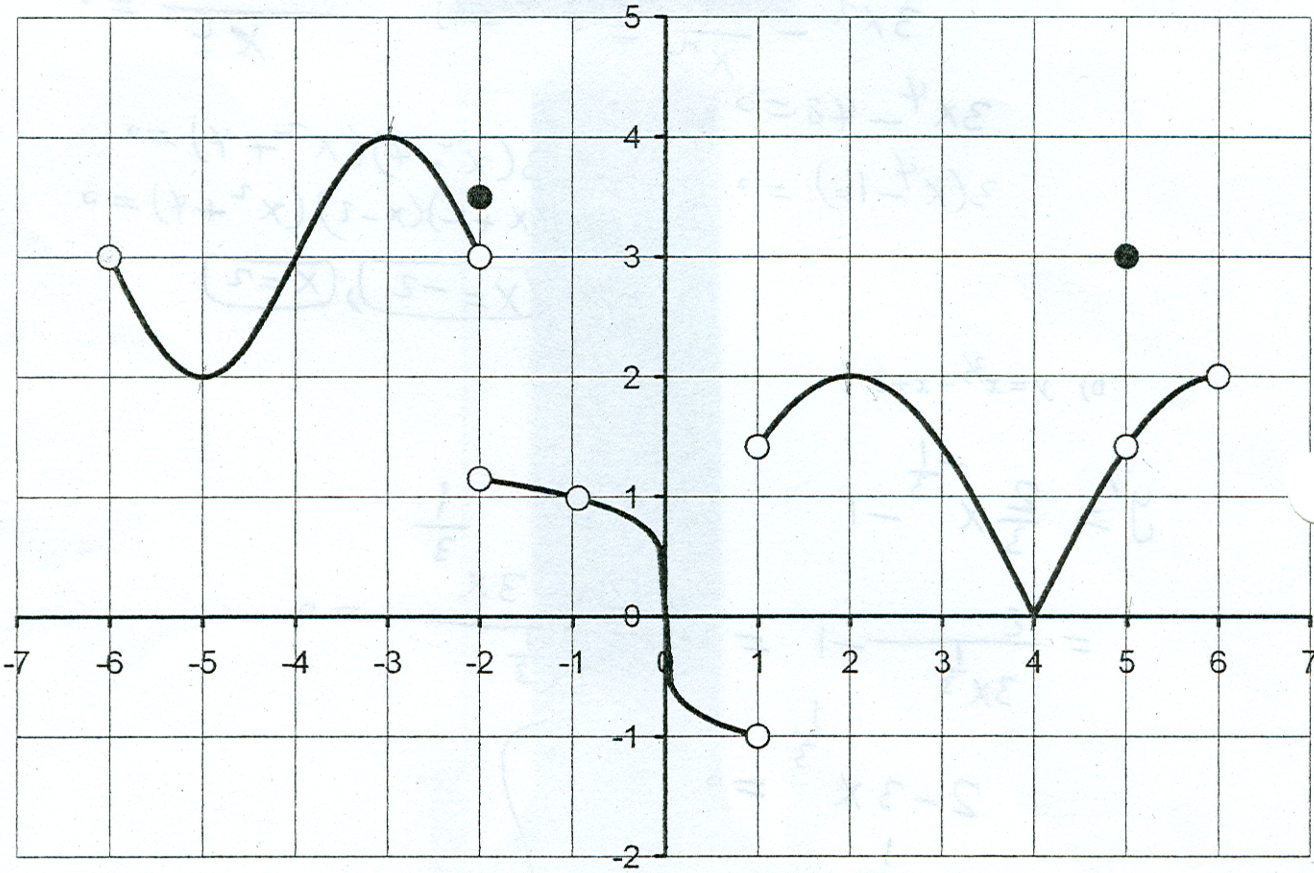
C.P $\Rightarrow$ $\boxed{x = \frac{8}{27}}$; $\boxed{x = 0}$

12) (10 points)

The graph below is the graph of a function f.
Assume f is undefined outside of (-6,6)

(Note: all critical numbers occur at integers, but some integers may not be critical numbers.)

For the graph above, list all critical numbers in (-6,6).



$\boxed{x=-5}$ $\boxed{x=-3}$

2pts
$\boxed{x=-2}$

2pts
$\boxed{x=0}$

$\boxed{x=2}$

$\boxed{x=4}$

2pts
$\boxed{x=5}$

13) Given that the function $f(x) = x^3 + ax^2 + bx + 1000$ has critical numbers at $x = -1$, and $x = 1$, find a and b. (10 points)

$f'(x) = 3x^2 + 2ax + b$

$f'(-1) = 3 - 2a + b = 0 \Rightarrow \boxed{3 - 2a + b = 0}$

$f'(1) = 3 + 2a + b \longrightarrow \boxed{3 + 2a + b = 0}$ ADD

$6 + 2b = 0$

$\boxed{b = -3}$

But; $3 - 2a + b = 0$

$3 - 2a - 3 = 0 \Rightarrow \boxed{a = 0}$

14) Find all value(s) of c (if any) that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = \dfrac{1}{1+x}$ on the interval $[0,1]$. (10 points)

$f(x) = (1+x)^{-1}$

$f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{\frac{1}{2} - 1}{1} = \dfrac{-\frac{1}{2}}{1} = -\frac{1}{2}$

$f'(x) = -1(1+x)^{-2} = \dfrac{-1}{(1+x)^2}$

$f'(x) = \dfrac{-1}{(1+x)^2}$ Must equal $-\frac{1}{2}$

$\therefore \quad \dfrac{-1}{(1+x)^2} = \dfrac{-1}{2} \Rightarrow \boxed{(1+x)^2 = 2}$

$1 + 2x + x^2 = 2$

$x^2 + 2x - 1 = 0$

$\boxed{x = 0.414}$ or $x = -2$

Not $[0$

$(1+x)^2 = 2$

$1 + x = \pm\sqrt{2}$

$-1 \qquad -1$

$x = \pm\sqrt{2} - 1$

15) Find the x-coordinate and the y-coordinate of relative maximum (or minimum) and the inflection point(s) of the function $f(x) = x^3 - x^2 - x + 1$.

(10 points)

$f'(x) = 3x^2 - 2x - 1$ $\quad f''(x) = 6x - 2$

$f'(x) = 0 \Rightarrow x = -\frac{1}{3}$ OR $x = 1$

$y = 1.185$ $\quad y = 0$

since $f''(-\frac{1}{3}) = 6(-\frac{1}{3}) - 2 = -2 - 2 = -4 \xRightarrow{CD} \boxed{(-\frac{1}{3}, 1.185)} =$

Since $f''(1) = 6(1) - 2 = 4$ CU $\Rightarrow \boxed{(1,0)}$ is a Relative

$f''(x) = 6x - 2 = 0$ $\quad \boxed{x = \frac{2}{6} = \frac{1}{3};\ y = 0.59}$

$f''(x) = -$ $\quad f''(x) = +$

$\frac{1}{3}$

Inflection Point.

16) A company has cost function $C(x) = 100 - 14x + x^2$ and demand function $p(x) = 18 - x$, where x is the number of staplers and $p(x)$ is in dollars.

a) How many units should the company make to maximize its profit? (5 points)

$R(x) = x(18 - x) = 18x - x^2$

$P(x) = (18x - x^2) - (100 - 14x + x^2) = 18x - x^2 - 100 + 14x - x^2$

$= -2x^2 + 32x - 100$

$P'(x) = -4x + 32 = 0 \Rightarrow \boxed{x = 8 \text{ units}}$

b) How much is the maximum profit?

(5 points)

$P(8) = -2(8)^2 + 32(8) - 100 = \$28$