Calculus I Test V Professor: Fred Katiraie Date: December 12, 2006

Name: Solution (1 Point)

Total Possible Points = 140 + 9 points Extra Credit

This is not a graphing calculator test.
I will not give credit to answers not supported by your work
No Procedure = No Points

1) Find the derivative of the function $y = (2x)^{\cos x}$ **(7 points)**

Compute y' in terms of x (Hint: Use Natural Logarithms)

$\ln y = \cos x \ln(2x)$

$\frac{1}{y} y' = -\sin x \ln 2x + \cos x \cdot \frac{1}{2x} \cdot 2$

$y' = y\left[-\sin x \ln 2x + \frac{\cos x}{x}\right] = \left[\sin x \ln 2x + \frac{\cos x}{x}\right] \cdot (2x)^{\cos x}$

2) Show that the following curves are orthogonal (i.e Perpendicular) **(8 Points)**

$2x^2 + y^2 = 3$ $\quad 4x + 2yy' = 0 \Longrightarrow y' = \frac{-4x}{2y} = \frac{-2x}{y}$

$x = y^2$ $\quad 1 = 2yy' \Longrightarrow y' = \frac{1}{2y}$

At Intersection Point

$2x^2 + x = 3$

$2x^2 + x - 3 = 0$

$(2x+3)(x-1) = 0$

~~$x = -\frac{3}{2}$~~ $\quad x = 1$ then $y = \pm 1$

at $(1,1)$ $\quad y_1' = \frac{-2}{1} = -2$; $y_2' = \frac{1}{2}$

at $(1,-1)$ $\quad y_1' = \frac{-2}{-1} = 2$; $y_2' = \frac{-1}{2}$

Since these slopes are Negative Reciprocal Hence, these curves are orthogonal

3) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1,0) (10 points)

$D = \sqrt{(y-0)^2 + (x-1)^2}$

$D^2 = y^2 + (x-1)^2$

$D^2 = 4 - 4x^2 + (x-1)^2$

$2D\frac{dD}{dx} = -8x + 2(x-1) = 0$

$\rightarrow -8x + 2x - 2 = 0$

$-6x - 2 = 0$

$-6x = 2$

$x = \frac{-1}{3}$;

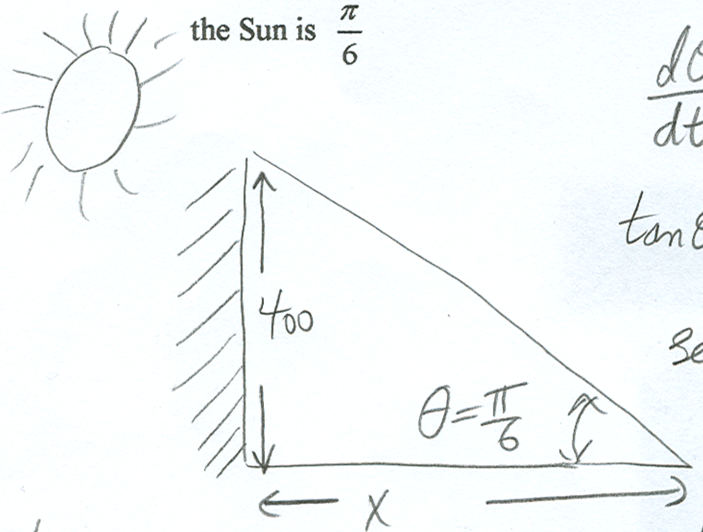
$y^2 = 4 - 4x^2$

$y^2 = 4 - 4\left(\frac{-1}{3}\right)^2$

$= 4 - \frac{4}{9} = \frac{32}{9}$

$x = \frac{-1}{3}$; $y = \pm \frac{4\sqrt{2}}{3}$

4) The angle of elevation of the Sun is decreasing at a rate of 0.25 rad/hour. How fast is the shadow cast by a 400–foot–tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{6}$ (7 points)



$$\frac{d\theta}{dt} = -0.25 \frac{rad}{HR}$$

$$\tan\theta = \frac{400}{X} = 400X^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = -400X^{-2}\frac{dX}{dt}$$

when $\theta = \frac{\pi}{6}$

$$\tan\frac{\pi}{6} = \frac{400}{X} \Rightarrow X = \frac{400}{\tan\frac{\pi}{6}} = 692.8 \text{ feet}$$

$$\left(\frac{1}{\cos\frac{\pi}{6}}\right)^2(-0.25) = \frac{-400}{692.8^2}\frac{dX}{dt}$$

$$\Rightarrow \boxed{\frac{dX}{dt} = 400\frac{ft}{HR}}$$

5a) Given $f'(x) = \sqrt{x} \bullet (6+5x)$ and f(1) = 10; Find $f(x)$ (4 points)

$$f'(x) = 6X^{1/2} + 5X^{3/2}$$

$$f(x) = \frac{6X^{3/2}}{\frac{3}{2}} + \frac{5X^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$10 = 4 + 2 + C$$

$$\boxed{4 = C}$$

$$\boxed{f(x) = 4X^{\frac{3}{2}} + 2X^{\frac{5}{2}} + 4}$$

5b) Given $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, $f(2) = 0$; Find $f(x)$ (4 points)

$$f'(x) = -X^{-1} + C$$

$$f(X) = -\ln X + CX + D$$

$$f(1) = 0 = -\ln(1) + C(1) + D \Rightarrow C + D = 0 \Rightarrow C = -D$$

$$f(2) = 0 = -\ln 2 + 2C + D$$

$$0 = -\ln 2 + 2(-D) + D$$

$$\boxed{D = -\ln 2} \quad \boxed{C = \ln 2}$$

$$\boxed{f(x) = -\ln X + (\ln 2)X - \ln 2}$$

$$\frac{d(\cot x)}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

5) Find the most general antiderivative of the following functions. (5 points each)

a. $f(x) = 9x^{\frac{3}{4}} + \csc^2 x + a^x$

$$F(x) = \frac{9x^{\frac{7}{4}}}{\frac{7}{4}} - \cot x + \frac{a^x}{\ln a} + c = \frac{36}{7}x^{\frac{7}{4}} - \cot x + \frac{a^x}{\ln a} + c$$

b. $f(x) = (3 - \frac{2}{\sqrt{x}})(3 + \frac{2}{\sqrt{x}}) = 9 - \frac{4}{x}$

$$F(x) = 9x - 4\ln|x| + c$$

c. $f(t) = \frac{t^5 + 2t^4}{\sqrt{t}} \bullet \frac{2}{\sqrt{t}} = \frac{2t^5 + 4t^4}{t} = 2t^4 + 4t^3$

$$F(t) = \frac{2t^5}{5} + \frac{4t^4}{4} + c = \frac{2}{5}t^5 + t^4 + c$$

d. $f(x) = \pi^3 x + e^4$

$$F(x) = \pi^3 \frac{x^2}{2} + e^4 x + c$$

e. $f(t) = -3\sec t \tan t + 2\sec^2 t$

$$F(t) = -3\sec t + 2\tan t + c$$

f. $f(t) = \frac{3}{\sqrt{1-t^2}} - \frac{7}{t^2+1}$

$$F(t) = 3\sin^{-1} t - 7\tan^{-1} t + c$$

6) Suppose we wish to estimate the area under the graph of $f(x)=16x-x^3$ for $1 \le x \le 5$. What is the value of the estimate using 7 approximating rectangles and making sample points to be the midpoints? (5 points)

$\Delta X = \frac{5-1}{7} = \frac{4}{7}$

$\text{sum}(\text{seq}((16x-x^3)*\frac{4}{7}, x, 1+\frac{4/7}{2}, 5-\frac{4/7}{2}, \frac{4}{7}) = 53.748 \text{ unit}^2$

7) Estimate the area under the graph of $f(x)=\frac{1}{x}+16x$ from $x=3$ to $x=11$ using 6 rectangles and the left endpoints. (5 points)

$\Delta X = \frac{11-3}{6} = \frac{8}{6} = \frac{4}{3}$

$\text{sum}(\text{seq}((\frac{1}{x}+16x)*\frac{4}{3}, x, 3, 11-\frac{4}{3}, \frac{4}{3})) = 812.143 \text{ units}^2$

8) If $\int_0^5 f(x)dx = 13$, $\int_3^5 f(x)dx = 4$, and $\int_2^5 f(x)dx = 9$,

Find the value of $\int_0^2 f(x)dx - \int_3^0 f(x)dx$ (5 points)

$\int_0^2 f(x)dx = 13-9 = 4$

$\int_0^3 f(x)dx = 13-4 = 9$

$\Rightarrow \int_0^2 f(x)dx - \int_3^0 f(x)dx = 4 + \int_0^3 f(x)dx$

$= 4 + 9 = 13$

9) If $\int_0^3 f(x)dx = 21$, and $\int_0^3 g(x)dx = 4$,

Find the value of $\int_0^3 f(x) * g(x)dx$ (5 points)

Can not be evaluated based on the Given Information

10) If $\int_0^3 f(x)dx = 12$, and $\int_0^6 f(x)dx = 55$,

Find the value of $\int_6^3 (-3f(x) + x^2 + 20)dx$ (5 points)

$$\int_3^6 f(x)dx = \int_0^6 f(x)dx - \int_0^3 f(x)dx = 55 - 12 = 43$$

$$\int_6^3 f(x)dx = -43$$

$$\int_6^3 (x^2+20)dx = \frac{x^3}{3} + 20x \Big|_6^3 = \left(\frac{27}{3} + 60\right) - \left(\frac{6^3}{3} + 20(6)\right) = -123$$

$$\int_6^3 (-3f(x) + x^2 + 20)dx = -3(-43) + -123 = 6$$

11) The velocity of a particle moving along a line is $t^2 - 3t - 4$ meters per second.

a) Find the displacement of the particle during time period $1 \le t \le 8$ seconds. (5 points)

$$\int_1^8 (t^2 - 3t - 4)dt = 47.8\overline{3} \text{ meters}$$

b) Find the distance traveled during the time period $1 \le t \le 8$ seconds. (5 points)

$$\left|\int_1^4 (t^2 - 3t - 4)dt\right| + \left|\int_4^8 (t^2 - 3t - 4)dt\right|$$

$$= 13.5 + 61.\overline{33} = 74.8\overline{3} \text{ meters}$$

12) Let $f(x)=\frac{1}{2}\int_{2x}^{5x} t^3 dt$

Find the value of $f'(x)$ (10 points)

$$f(x)=\frac{1}{2}\left[\int_{2x}^{c} t^3 dt + \int_{c}^{5x} t^3 dt\right] = \frac{1}{2}\left[-\int_{c}^{2x} t^3 dt + \int_{c}^{5x} t^3 dt\right]$$

$$f'(x)=\frac{1}{2}\left[-(2x)^3(2)+(5x)^3(5)\right]$$

$$=\frac{1}{2}\left[-16x^3+625x^3\right] = \boxed{\frac{609}{2}x^3}$$

13) Given $\int_a^b x dx = 30$ and $\int_b^a 5dx = -30$ Find a and b. (8 points)

$$\left.\frac{x^2}{2}\right|_a^b = 30 \Rightarrow \frac{b^2}{2}-\frac{a^2}{2}=30 \Rightarrow \boxed{b^2-a^2=60} \text{ (1)}$$

$$\int_b^a 5dx = 5x\Big|_b^a = 5a-5b = -30 \Rightarrow a-b=-6 \Rightarrow \boxed{b=a+6} \text{ (2)}$$

$$b^2-a^2=60 \Rightarrow (a+6)^2-a^2=60$$

$$a^2+12a+36-a^2=60 \Rightarrow \boxed{a=2} \text{ and } \boxed{b=8}$$

14) Verify by differentiation that the following formula is correct.

$$\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$$ (5 points)

$$\left[\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c\right]' = \frac{1}{2a}\cdot\frac{1}{\frac{u-a}{u+a}}\cdot\frac{1(u+a)-1(u-a)}{(u+a)^2}+0$$

$$=\frac{1}{2a}\cdot\frac{u+a}{u-a}\cdot\frac{2a}{(u+a)^2} = \boxed{\frac{1}{u^2-a^2}}$$

Yes, the formula is correct!!

15) Find the value of the integral $\int_C^D \frac{x^2-1}{x}dx$ (5 points)

(Assume C >0 and D > 0, and leave your answer in terms of C and D)

$$\int_C^D \frac{x^2-1}{x}dx = \int_C^D \left(\frac{x^2}{x} - \frac{1}{x}\right)dx = \int_C^D \left(x - \frac{1}{x}\right)dx = \frac{x^2}{2} - \ln x \Big|_C^D$$

$$= \left(\frac{D^2}{2} - \ln D\right) - \left(\frac{C^2}{2} - \ln C\right)$$

$$= \frac{D^2}{2} - \ln D - \frac{C^2}{2} + \ln C$$

16) Find the area enclosed by the following curves: (7 points)

$y = 2x - x^2$

and

$y = 2x - 4$

$$2x - x^2 = 2x - 4 \Rightarrow x^2 = 4$$

$$x = \pm 2$$

$$\int_{-2}^{2}\left[(2x - x^2) - (2x-4)\right]dx = \int_{-2}^{2}(-x^2+4)\,dx = 10.67 \text{ unit}^2$$

Extra Credit

17) Water flows from the bottom of a storage tank at a rate of

$r(t) = 200 - 4t \ \frac{\text{liters}}{\text{minute}}$, Where $0 \le t \le 50$ minutes (3 points each)

a) Find the amount of water that flows from the tank initially (at time t = 0).

$\int_0^0 (200-4t)\,dt = 0$ liters

b) Find the amount of water that flows from the tank during the first 10 minutes.

$\int_0^{10} (200-4t)\,dt = 1800$ liters

c) Find the amount of water that flows from the tank during the first 25 minutes.

$\int_0^{25} (200-4t)\,dt = 3750$ liters