Calculus I   Test V Professor: Fred Katiraie   May 2006

Name: Key

(1) Let $y = f(x)$ be implicitly defined as $x^{\cos y} = y^{\sin x}$

Compute y' in terms of x, and y. (Hint: Use Natural Logarithms) (10 points)

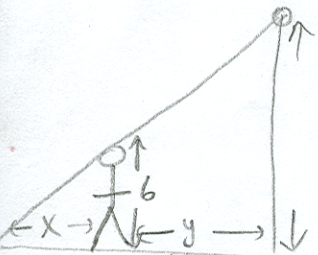
$$\cos y \ln x = \sin x \ln y$$

$$-\sin y \, y' \ln x + \cos y \frac{1}{x} = \cos x \ln y + \sin x \frac{1}{y} y'$$

$$y'\left[-\sin y \ln x - \frac{\sin x}{y}\right] = \cos x \ln y - \frac{\cos y}{x}$$

$$\boxed{y' = \frac{\cos x \ln y - \frac{\cos y}{x}}{-\sin y \ln x - \frac{\sin x}{y}}} = \frac{\frac{x \cos x \ln y - \cos y}{x}}{\frac{-y \sin y \ln x - \sin x}{y}} = \boxed{\frac{y(x \cos x \ln y - \cos y)}{x(-y \sin y \ln x - \sin x)}}$$

(2) A man 6 ft tall walks at a rate of 1.5 ft/sec away from a lamppost that is 20 feet high. At what rate is the length of his shadow changing when he is 50 feet away from the lamppost. (10 points)



$\frac{dy}{dt} = 1.5 \text{ft/sec}$

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$$\frac{dx}{dt} = ?$$

$$\frac{6}{x} = \frac{20}{x+y}$$

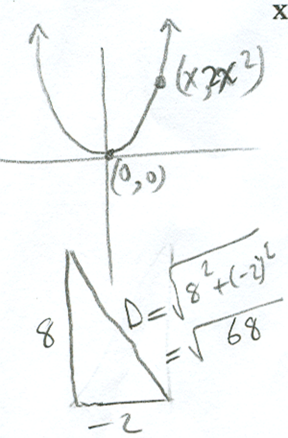
$$6x + 6y = 20x$$

$$6y = 14x$$

then $x = \frac{6}{14} y$

$$\frac{dx}{dt} = \frac{6}{14}\left(\frac{dy}{dt}\right) \longrightarrow \frac{dx}{dt} = \frac{6}{14}(1.5) = \frac{9}{14} \boxed{= 0.643 \frac{\text{ft}}{\text{sec}}}$$

(3) A particle starts at the origin and moves along the parabola $y = 2x^2$ such that its distance from the origin increases at 7 meters per second. How fast is the particle's x-coordinate changing as it passes through the point (-2,8)? (10 points)



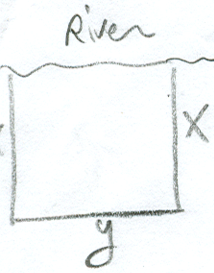
$$D = \sqrt{(x-0)^2 + (2x^2-0)^2} \Rightarrow D = \sqrt{x^2 + 4x^4}$$

$$D^2 = x^2 + 4x^4$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 16x^3\frac{dx}{dt} \Rightarrow 2D\frac{dD}{dt} = \frac{dx}{dt}\left[2x + 16x^3\right]$$

$$\frac{dx}{dt} = \frac{2D\frac{dD}{dt}}{2x+16x^3} = \frac{2(\sqrt{68})(7)}{2(-2)+16(-2)^3} = \boxed{\approx -0.875 \text{ m/sec}}$$

(4) A rancher intends to fence off three sides of a rectangular region along a straight stretch of river (no fence along the river). The enclosed area is to be 3600 square meters.



a) Find an equation for the total length of fencing material, F(x), where x represents the lengths of the sides that are perpendicular to the river. (5 points)

$$2x + y = F(x) \quad \text{and} \quad x \cdot y = 3600 \Rightarrow y = \frac{3600}{x}$$

$$\boxed{F(x) = 2x + \frac{3600}{x}} \quad \text{(meters)}$$

b) What is the least amount of fencing material that is needed? (5 points)

$$F(x) = 2x + 3600x^{-1}$$

$$F'(x) = 2 - \frac{3600}{x^2} = 0 \Rightarrow 2x^2 = 3600 \Rightarrow x^2 = 1800$$

$$\boxed{x = 42.43 \text{ meters}}$$

then, F(42.43) = 169.71 meters is needed for fencing material

(5) Find the most general antiderivative of the following functions.

a) $f(x) = 12x^{\frac{3}{4}} + 6x^{\frac{1}{3}} - 5$ (3 points)

$F(x) = \frac{12x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} - 5x + c = \frac{48}{7}x^{\frac{7}{4}} + \frac{9}{2}x^{\frac{4}{3}} - 5x + c$

b) $f(x) = (5 - \frac{2}{\sqrt{x}})(5 + \frac{2}{\sqrt{x}}) = 25 - \frac{4}{x}$ (2 points)

$F(x) = 25x - 4\ln|x| + c$

c) $f(t) = \frac{4t^8 + 2t^2}{\cancel{\sqrt{t}}} \bullet \frac{\cancel{\sqrt{t}}}{2} = 2t^8 + t^2$ (3 points)

$F(t) = \frac{2t^9}{9} + \frac{t^3}{3} + c$

d) $f(x) = \pi^x + e\,x$ (2 points)

$F(x) = \frac{\pi^x}{\ln \pi} + \frac{ex^2}{2} + c$

e) $f(t) = e^t + \sec t \tan t + 2$ (5 points)

$F(t) = e^t + \sec t + 2t + c$

f) $f(t) = \frac{10}{t^2+1} + \frac{9}{\sqrt{1-t^2}}$ (5 points)

$F(t) = 10\tan^{-1}(t) + 9\sin^{-1}(t) + c$

The acceleration of an object dropped or thrown on Earth is -32 $\frac{feet}{sec^2}$

$S(0) = 50$ feet

$V(0) = 20$ ft/sec

(6) A ball is thrown directly upward at a speed of 20 feet per second from a cliff 50 feet above the ground. (2 points each)

g) Find expressions for the velocity and height of the ball t seconds after it was released.

$V(t) = -32t + C$ But $V(0) = 20 \Rightarrow$ $\boxed{V(t) = -32t + 20}$ ft/sec

$\boxed{S(t) = -16t^2 + 20t + 50 \text{ ft}}$

h) At what time does the ball reach its highest point?

$V(t) = 0 \Rightarrow t = \frac{20}{32} = \frac{5}{8} = 0.625$ Seconds

i) How high above the ground (from the base of the cliff) does the ball reach?

$S(0.625) = -16(0.625)^2 + 20(0.625) + 50 = 56.25$ ft

j) When does the ball strike the ground at the base of the cliff?

$S(t) = 0 \Rightarrow t = 2.5$ Seconds

k) What is its velocity at that instant (i.e. when the ball hits the ground)?

$V(2.5) = -32(2.5) + 20 = -60$ ft/sec

(7) Suppose we wish to estimate the area under the graph of $f(x) = 16 - 2x^2$ for $0 \le x \le 6$. What is the value of the estimate using 12 approximating rectangles and making sample points to be midpoints? (5 points)

$\Delta x = \frac{6-0}{12} = 0.5$

Area = sum(seq($(16-2x^2) * 0.5, x, 0 + \frac{0.5}{2}, 6 - \frac{0.5}{2}, 0.5$) = 47.75 units^2

(8) Estimate the area under the graph of $f(x) = 16x - \frac{1}{x^2}$ from $x = 1$ to $x = 5$ using four rectangles and left endpoints. (5 points)

$\Delta x = \frac{5-1}{4} = 1$

Area = Sum(seq($(16x - \frac{1}{x^2}) * 1, x, 1, 5-1, 1$) = 158.6 units^2

(9) If $\int_0^3 f(x)dx = 14$, $\int_3^6 f(x)dx = 4$, and $\int_2^6 f(x)dx = 15$,

find the value of $\int_0^2 f(x)dx$ (5 points)

$$\int_0^2 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx - \int_2^6 f(x)\,dx$$

$$= 14 + 4 - 15 = \boxed{3}$$

(10) If $\int_0^3 f(x)dx = 7$, and $\int_0^3 g(x)dx = 4$,

find the value of $\int_0^3 \frac{f(x)}{g(x)}dx$ undefined (5 points)

Can not be done given the given Information.

(11) If $\int_3^0 f(x)dx = 11$, and $\int_0^6 f(x)dx = -55$,

Find the value of $\int_3^6 (3f(x) - x - 20)dx$ (10 points)

$$\int_0^3 f(x)\,dx = -11 \text{ and } \int_0^6 f(x)\,dx = -55 \Rightarrow \int_3^6 f(x)\,dx = \int_0^6 f(x)\,dx - \int_0^3 f(x)\,dx$$

$$= -55 - (-11) = -44$$

$$3\int_3^6 f(x)\,dx = 3(-44) = \boxed{-132}$$ 4pts

$$\int_3^6 -x\,dx = -\frac{x^2}{2}\Big|_3^6 = -\frac{1}{2}(6^2 - 3^2) = -\frac{1}{2}(36 - 9) = \boxed{-13.5}$$ 3pts

$$\int_3^6 -20\,dx = -20(6-3) = -20(3) = \boxed{-60}$$ 3pts

$$\int_3^6 (3f(x) - x - 20)dx = -132 - 13.5 - 60 = -205.5$$

(12) Find the value of the integral $\int \frac{x^2-1}{x-1}dx$

(5 points)

$$\int \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}\,dx = \frac{x^2}{2} + x + C$$

(13) A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 25$ (measured in meters per second).

a) Find the displacement of the particle during time period $1 \le t \le 8$ seconds.

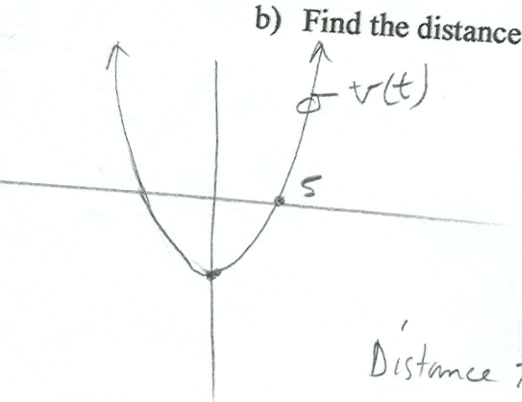
(5 points)

$$\int_1^8 (t^2-25)\,dt = \boxed{-4.67} \text{ meters}$$

(i.e 4.67 meters to the left of the starting point.)

b) Find the distance traveled during the time period $1 \le t \le 8$ seconds.

(5 points)



$$\text{Distance travelled} = \int_1^5 v(t)\,dt + \int_5^8 v(t)\,dt$$

$$= \left|\int_1^5 (t^2-25)\,dt\right| + \left|\int_5^8 (t^2-25)\,dt\right|$$

$$= 58.67 + 54$$

$$= \boxed{112.67 \text{ meters}}$$

(14) Find the area enclosed by the following curves:

$f(x) = 2x - x^2$
$g(x) = 2x - 4$ (5 points)

$$\int_{-2}^{2} (2x - x^2) - (2x - 4)\, dx = \int_{-2}^{2} (-x^2 + 4)\, dx = 10.67 \text{ units}^2$$

(15) Let $f(x) = -\int_{2x}^{6x} t^3 dt$

Find the value of $f'(2)$ (5 points)

$$-\left[\int_{2x}^{0} t^3\, dt + \int_{0}^{6x} t^3\, dt\right]$$

$$-\left[-\int_{0}^{2x} t^3\, dt + \int_{0}^{6x} t^3\, dt\right] = \int_{0}^{2x} t^3\, dt - \int_{0}^{6x} t^3\, dt$$

$$f'(x) = (2x)^3(2) - (6x)^3(6) = 16x^3 - 1296x^3 = -1280x^3 \quad f'(2) = -10240$$

(16) If $F(x) = \int_{2}^{3x} (\ln(t) + 6t + 2)dt$, find the value of $F'(x)$. (5 points)

$$F'(x) = [\ln(3x) + 6(3x) + 2] \cdot 3$$

$$\boxed{F'(x) = 3\ln(3x) + 54x + 6}$$

(17) Determine whether the following integral formula is correct. Not Correct

$$\int\left(\frac{x}{\sqrt{1+x^2}}+5x\right)dx=\sqrt{1+x^2}+5x+C$$

~~(5 points)~~

$$\left(\sqrt{1+x^2}+5x+c\right)' = \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}\cdot\cancel{2}x+5 = \frac{x}{\sqrt{1+x^2}}+5$$

(18) Evaluate the following integral (5 points)

$$\int(\sin\theta(\cot\theta+\csc\theta)d\theta = \int \sin\theta\cdot\left(\frac{\cos\theta}{\cancel{\sin\theta}}+\frac{1}{\cancel{\sin\theta}}\right)d\theta = \int(\cos\theta+1)\,d\theta$$

$$\boxed{= \sin\theta+\theta+C}$$

(19)Bonus Question: Not Correct (10 points)

Determine by differentiation ~~whether the following integral formula~~ is correct.

$$\int\frac{3du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$$

Because;

$$\left[\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c\right]' = \frac{1}{2a}\,\frac{1}{\frac{u-a}{u+a}}\cdot\frac{1(u+a)-1(u-a)}{(u+a)^2}+0$$

$$= \frac{1}{\cancel{2a}}\,\frac{\cancel{u+a}}{u-a}\cdot\frac{\cancel{2a}}{(\cancel{u+a})(u+a)} = \frac{1}{(u-a)(u+a)}$$

$$\boxed{=\frac{1}{u^2-a^2}}$$