Professor Katiraie   Calculus I   Fall 2006   Test IV (Chapters 3, and 4)

Name:____________________(1 Point)   Total Possible Points = 140 (Plus 10 pts Extra Credits ☺)

Show All Your Work,   **No Procedure = No Points**

(7 points) **1)** Find the derivative of the function $y = (2x)^{\cos x}$ __

Compute y' in terms of x (Hint: Use Natural Logarithms)

$$\ln y = \cos x \ln(2x)$$

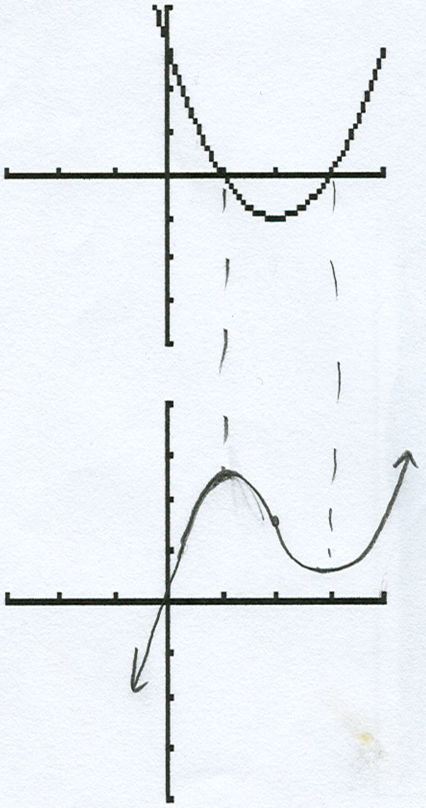
$$\frac{1}{y} y' = -\sin x \ln(2x) + \cos x \frac{1}{2x} \cdot 2$$

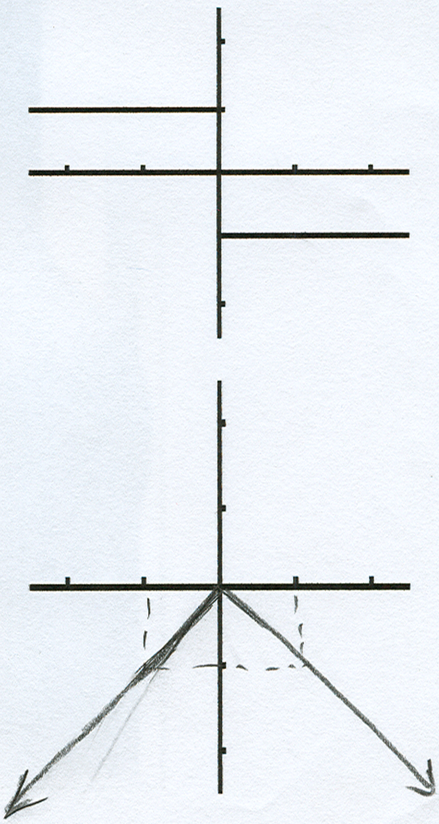
$$\boxed{y' = (2x)^{\cos x}\left(-\sin x \ln(2x) + \frac{\cos x}{x}\right)}$$

(6 Points)

2) Given the graphs of $y = f'(x)$,

Sketch the graphs of $y = f(x)$





3) Given the following ellipse $2x^2 + y^2 = 1$.

a) At what point(s) is the slope of the tangent line equal to 1? **(5 Points)**

$$4x + 2yy' = 0 \Rightarrow y' = \frac{-4x}{2y} = \frac{-2x}{y} = 1 \Rightarrow \boxed{y = -2x}$$

$$2x^2 + (-2x)^2 = 1$$

$$2x^2 + 4x^2 = 1 \Rightarrow 6x^2 = 1 \quad \left(x = \pm\sqrt{\tfrac{1}{6}} \;;\; y = \mp 2\sqrt{\tfrac{1}{6}}\right)$$

b) At what point(s) is the slope of the tangent line equal to 0? **(5 Points)**

$$\frac{-2x}{y} = 0 \Rightarrow x = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

$$\boxed{\begin{matrix}(0,1)\\(0,-1)\end{matrix}}$$

4) Show that the following curves are orthogonal (i.e Perpendicular) **(9 Points)**

$$\begin{cases} 2x^2 + y^2 = 3 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = \frac{-4x}{2y} = \boxed{\frac{-2x}{y}} \\ x = y^2 \Rightarrow 1 = 2yy' \Rightarrow \boxed{y' = \frac{1}{2y}} \end{cases}$$

Intersection

$$2x^2 + x = 3$$

$$2x^2 + x - 3 = 0$$

$$(2x+3)(x-1) = 0$$

$$\cancel{x = -3/2} \quad x = 1$$

$\rightarrow$ If $x = 1$; $y = \pm 1$

$$\boxed{\begin{matrix} ⓐ\ (1,1) & \frac{-2(1)}{1} = -2 \;;\; \frac{1}{2(1)} = \frac{1}{2} \\ ⓐ\ (1,-1) & \frac{-2(1)}{-1} = 2 \;;\; \frac{1}{2(-1)} = -\frac{1}{2} \end{matrix}}$$

Hence curves are orthogonal

5) Find the equation of the tangent line to the parametric curve
$x = t^2 + 3, \quad y = 2t^3 - t$ at the point corresponding to $t = 2$. **(8 Points)**

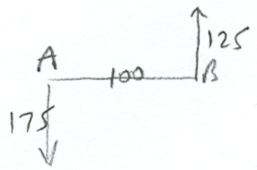
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{6t^2-1}{2t}\right|_{t=2} = \frac{6(2)^2-1}{2(2)} = \frac{23}{4}$$

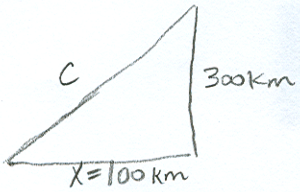
$$x = 2^2 + 3 = 7 \qquad y = 2(2)^3 - 2 = 16 - 2 = 14$$

$$y - 14 = \frac{23}{4}(x - 7) \Rightarrow \boxed{y = \frac{23}{4}x - \frac{105}{4}}$$

For Problems 6 - 8: Pick two of the following three problems. Clearly indicate the two problems you want to be graded.

6) At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 5:00 PM.? (10 points)





$$c^2 = x^2 + y^2$$

$$\cancel{2}c\frac{dc}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

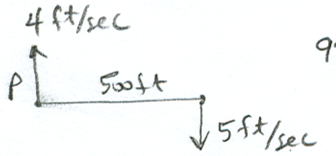
$$0$$

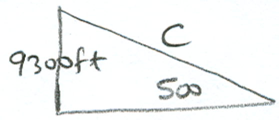
$$\frac{dc}{dt} = \frac{(300)(60)}{316.23\text{Km}} = \boxed{56.92\ \frac{\text{Km}}{\text{HR}}}$$

$$C = \sqrt{100^2 + 300^2}$$

$$c = 316.23\ \text{Km}$$

7) A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 minutes after the woman starts walking? (10 points)





$$c^2 = x^2 + y^2$$

$$\cancel{2}c\frac{dc}{dt} = 2x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$0$$

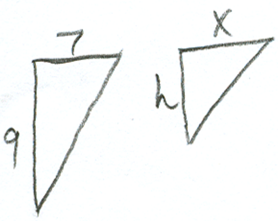
4ft/sec * 20 * 60 = 4800ft

5ft/sec * 15*60 = 4500ft

$$C = \sqrt{500^2 + 9300^2} = 9313.43$$

$$\frac{dc}{dt} = \frac{9300(9)}{9313.43} = \boxed{8.99\ \text{ft/sec}}$$

8) A container in the shape of an inverted right circular cone has a radius of 7.00 inches at the top and a height of 9.00 inches. At the instant when the water in the container is 5.00 inches deep, the surface level is falling at the rate of - 0.700 in./sec. Find the rate at which water is being drained? (10 points)



$$\frac{7}{9} = \frac{x}{h} \Rightarrow x = \frac{7h}{9}$$

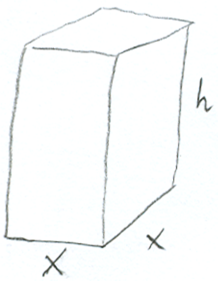
$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi\left(\frac{7}{9}h\right)^2 h = \frac{49}{243}\pi h^3$$

$$\frac{dV}{dt} = \frac{49\pi}{243}3h^2\frac{dh}{dt} = \frac{49\pi}{243}(3)(5)^2(-0.7) = \boxed{-33.26\ \frac{\text{in}^3}{\text{sec}}}$$

For problems 9 - 11: Pick two of the following three problems. Clearly indicate the two problems you want to be graded.

9) If 1200 sq. cm of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

(10 points)



$$x^2 + 4xh = 1200 \Longrightarrow h = \frac{1200 - x^2}{4x}$$

$$V = x^2 h = x^2\left(\frac{1200 - x^2}{4x}\right) = \frac{1200x^2}{4x} - \frac{x^4}{4x} = 300x$$

$$\frac{dV}{dx} = 300 - \frac{3}{4}x^2 = 0$$

$$\frac{3}{4}x^2 = 300 \Longrightarrow x^2 = 300 \cdot \frac{4}{3} = 400$$

$$x = 20\text{ cm} \quad h = 10\text{ cm} \quad \boxed{V_{max} = 4000\text{ cm}}$$

10) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1,0)

(10 points)

$$y^2 = 4 - 4x^2$$

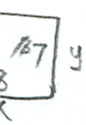
$$D = \sqrt{(x-1)^2 + (y-0)^2} \Longrightarrow D^2 = (x-1)^2 + y^2 \quad \text{But } y^2 = 4 - 4x^2$$

$$D^2 = (x-1)^2 + 4 - 4x^2 = x^2 - 2x + 1 + 4 - 4x^2 = -3x^2 - 2x + 5$$

$$D\frac{dD}{dx} = -6x - 2 \quad \text{let } \frac{dD}{dx} = 0 \text{ we get } -6x - 2 = 0 \; ; \; x = \frac{2}{-6} = \frac{-1}{3}$$

$$y = \pm\sqrt{4 - 4(-\tfrac{1}{3})^2} = \pm 1.$$

$$D = \sqrt{(\tfrac{-1}{3} - 1)^2 + (3.556)} = 2.31$$



11) A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$8 per foot for two opposite sites, and \$7 per foot for the other two sides. Find the dimensions of the field of area 730 sq. ft. that would be the cheapest to enclose.

(10 points)

$$2(8x) + 2(7y) = 16x + 14y \quad ; \quad A = x \cdot y = 730 \Longrightarrow y = \frac{730}{x}$$

$$= 16x + 14\left(\frac{730}{x}\right) = 16x + 10220x^{-1}$$

$$= 16 - 10220x^{-2} = 0$$

$$16 - \frac{10220}{x^2} = 0$$

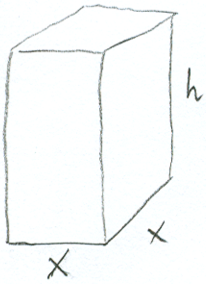
$$x^2 = \frac{10220}{16}$$

$$\boxed{x = 25.27\text{ ft}} \rightarrow \boxed{y = \frac{730}{25.27} = 28.9\text{ ft}}$$

∴ Dimension of the field is 25.27 ft by 28.9 ft

For problems 9 - 11: Pick two of the following three problems. Clearly indicate the two problems you want to be graded.

9) If 1200 sq. cm of material is available to make a box with a square base and an open top, find the largest possible volume of the box. (10 points)



$$x^2+4xh=1200 \Rightarrow h=\frac{1200-x^2}{4x}$$

$$V=x^2h=x^2\left(\frac{1200-x^2}{4x}\right)=\frac{1200x^2}{4x}-\frac{x^4}{4x}=300x-\frac{1}{4}x^3$$

$$\frac{dV}{dx}=300-\frac{3}{4}x^2=0$$

$$\frac{3}{4}x^2=300 \Rightarrow x^2=300\cdot\frac{4}{3}=400$$

$$x=20\text{ cm} \quad h=10\text{ cm} \quad \boxed{V_{max}=4000\text{ cm}^3}$$

10) Find the points on the ellipse $4x^2+y^2=4$ that are farthest away from the point (1,0) (10 points)

$$y^2=4-4x^2$$

$$D=\sqrt{(x-1)^2+(y-0)^2} \Rightarrow D^2=(x-1)^2+y^2 \quad \text{But } y^2=4-4x^2$$

$$\therefore D^2=(x-1)^2+4-4x^2=x^2-2x+1+4-4x^2=-3x^2-2x+5$$

$$2D\frac{dD}{dx}=-6x-2 \quad \text{let } \frac{dD}{dx}=0 \text{ we get } -6x-2=0; \quad x=\frac{2}{-6}=\frac{-1}{3}$$

$$y=\pm\sqrt{4-4(\tfrac{-1}{3})^2}=\pm1.89$$

$$D=\sqrt{(\tfrac{-1}{3}-1)^2+(3.556)}=2.31$$

11) A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$8 per foot for two opposite sites, and \$7 per foot for the other two sides. Find the dimensions of the field of area 730 sq. ft. that would be the cheapest to enclose. (10 points)

$$\text{Cost}=2(8x)+2(7y)=16x+14y \quad ; \quad A=x\cdot y=730 \Rightarrow y=\frac{730}{x}$$

$$\text{Cost}=16x+14\left(\frac{730}{x}\right)=16x+10220x^{-1}$$

$$C'=16-10220x^{-2}=0$$

$$16-\frac{10220}{x^2}=0$$

$$x^2=\frac{10220}{16}$$

$$\boxed{x=25.27\text{ ft}} \rightarrow \boxed{y=\frac{730}{25.27}=28.9\text{ ft}}$$

$\therefore$ Dimension of the field is 25.27 ft by 28.9 ft

12) Analytically find the exact value of all critical numbers of the following functions. (In other words, find the x-coordinates of the critical points.) (10 points)

a) $y = x^{\frac{4}{5}}(x-4)^2$

$y' = \frac{4}{5}x^{\frac{-1}{5}}(x-4)^2 + x^{\frac{4}{5}}(2)(x-4) = \frac{4(x-4)^2}{5(x)^{1/5}} + 2x^{\frac{4}{5}}(x-4)$

$y' = \frac{4(x-4)^2 + 10x^1(x-4)}{5x^{\frac{1}{5}}}$

$2(x-4)\left(2(x-4)+5x\right)=0$

$2x - 8 + 5x = 0$ $\quad +7x = +8$

ⓐ $\boxed{x=0}$ $f'(x)$ is undefined and ⓑ $\boxed{x=4}$ $\boxed{x = \frac{+8}{7}}$ $f'(x)=0$

$\boxed{\text{C.N are } x=0,\ x=\frac{+8}{7},\ x=4}$

b) $y = x^{\frac{2}{3}}(x^2-4)$

$y' = \frac{2}{3}x^{\frac{-1}{3}}(x^2-4) + x^{\frac{2}{3}}(2x)$

$f'(x) = \frac{2(x^2-4)}{3x^{\frac{1}{3}}} + 2x^{\frac{5}{3}} = \frac{2(x^2-4)+6x^2}{3x^{1/3}} = \frac{8x^2-8}{3x^{1/3}} = \frac{8(x^2-1)}{3x^{1/3}}$

$\boxed{\text{C.N are } x=0,\ x=\pm 1}$

13) Given that the function $f(x) = x^3 + ax^2 + bx + c$ has critical numbers at $x=-3$, and $x=2$, find a and b. (10 points)

$f'(x) = 3x^2 + 2ax + b$

$f'(-3) = 3(-3)^2 + 2a(-3) + b \Rightarrow \begin{cases} 27 - 6a + b = 0 \\ 12 + 4a + b = 0 \end{cases}$

$f'(2) = 3(2)^2 + 2a(2) + b \Rightarrow$

$\begin{cases} 27-6a+b=0 \\ -1 \quad 12+4a+b=0 \end{cases} \Rightarrow \begin{cases} 27-6a+b=0 \\ -12-4a-b=0 \end{cases}$

$15 - 10a = 0 \Rightarrow \boxed{a = \frac{15}{10} = \frac{3}{2} = 1.5}$

$12 + 4(1.5) + b = 0 \Rightarrow \boxed{b = -18}$

14) Find the extreme values of the function $f(x) = \dfrac{4x}{x^2+1}$ and where they occur

(Please Justify Your Answer Using Calculus Methods Discussed in Class) (10 points)

$$f'(x) = \frac{4(x^2+1) - 2x(4x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2} \quad X = \pm 1 \quad f'(x) = 0$$

$f'(x)$ $-$ | $f'(x)$ $+$ | $f'(x)$ $-$

$X=-1$, $y=-2$ Abs. Min

$X=1$, $y=2$ Abs. Max

15) A company has cost function $C(x) = 84 + 1.26x - 0.01x^2 + 0.00007x^3$ and demand function $p(x) = 3.5 - 0.01x$, where x is the number of staplers and $p(x)$ is in dollars.

a) How many units should the company make to maximize its profit? (5 points)

$$R(x) = X(3.5 - 0.01X) = 3.5X - 0.01X^2 \implies R'(x) = 3.5 - 0.02X$$

$$C'(x) = 1.26 - 0.02X + 3(0.00007)X^2 = 1.26 - 0.02X + 0.00021X^2$$

$R'(x) = C'(x)$ at Max. Profit

$$3.5 - \cancel{0.02X} = 1.26 - \cancel{0.02X} + 0.00021X^2 \rightarrow \boxed{X \approx 103 \text{ staples}}$$

b) How much is the maximum profit? (5 points)

Profit $=$ R $-$ C

$$P = X(3.5 - 0.01X) - (84 + 1.26X - 0.01X^2 + 0.00007X^3)$$

$\text{Profit}_{\text{max}} = $ (Profit function evaluated at $X = 103$) $\approx \$70.23$

16a) Given $f'(x) = \sqrt{x} \bullet (6+5x)$ and f(1) = 10; Find $f(x)$ (5 points)

$f'(x) = 6x^{1/2} + 5x^{3/2}$

$f(x) = \frac{6x^{3/2}}{\frac{3}{2}} + \frac{5x^{5/2}}{5/2} + C$

$\Rightarrow f(x) = 4x^{3/2} + 2x^{5/2} + C$

$f(1) = 10 \Rightarrow \boxed{C = 4}$

$\boxed{f(x) = 4x^{3/2} + 2x^{5/2} + 4}$

16b) Given $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, $f(2) = 0$; Find $f(x)$ (5 points)

$f'(x) = \frac{x^{-1}}{-1} + C$

$f(x) = -\ln x + Cx + D$

$\Rightarrow f(1) = 0 \Rightarrow 0 = -\ln(1) + C(1) + D \Rightarrow C = -D$

$f(2) = 0 \Rightarrow 0 = -\ln(2) + 2C + D$

$-\ln 2 + 2(-D) + D = 0$ $\boxed{D = \ln 2}$

$\boxed{C = \ln 2}$

$\boxed{f(x) = -\ln x + (\ln 2)x - \ln 2}$

16) A pumpkin pie is thrown upward with a speed of 48 ft/sec from the edge of a cliff 432 feet above the ground. (Assume gravity of earth is -32)

a) Find the pie's height above the ground t seconds later. (4 points)

$a = -32$

$v = -32t + C$ But $v(0) = 48$ ft/sec $\Rightarrow v(t) = -32t + 48$

$S(t) = -\frac{32t^2}{2} + 48t + C$ But $S(0) = 432 \Rightarrow \boxed{S(t) = -16t^2 + 48t + 432}$ feet

b) When does the pie reach its maximum height? (3 points)

When $v(t) = 0 \Rightarrow -32t + 48 = 0$ $\boxed{t = 1.5 \text{ seconds}}$

c) When does the pie hit the ground? (3 points)

$-16t^2 + 48t + 432 = 0 \Rightarrow -16(t^2 - 3t - 27) = 0$

$t = \frac{+3 \pm \sqrt{9 - 4(1)(-27)}}{2} = \frac{3 \pm 10.82}{2}$

$\rightarrow$ 6.91 seconds

$\rightarrow \cancel{-3.91}$ Ignore

Extra Credit Problems

17) Use Newton's Method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places (5 points)

let $Y_1 = \sin x - x^2 + 3x - 1$

$Y_2 = \cos x - 2x + 3$

Guess $x = 0.5$

0.5 store in x

$x - \frac{Y_1(x)}{Y_2(x)} \to x$ enter

0.2465

0.2687

0.268881

$\boxed{x = 0.268881}$

Guess $x = 3$

3 store in x

$x - \frac{Y_1(x)}{Y_2(x)} \to x$ Enter

2.784741

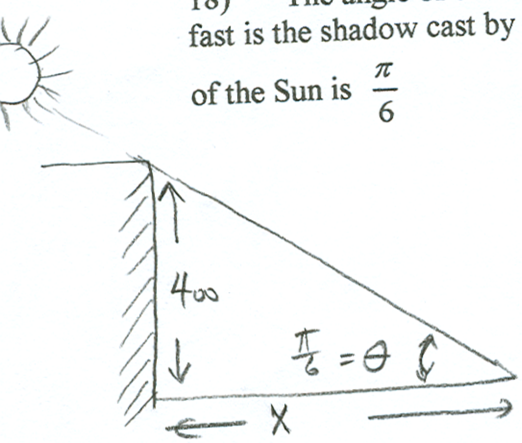
2.7701299

2.770057564

2.770057563

$\boxed{x = 2.770057563}$

18) The angle of elevation of the Sun is decreasing at a rate of 0.25 rad/hour. How fast is the shadow cast by a 400–foot–tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{6}$ (5 points)



$\frac{d\theta}{dt} = -0.25 \frac{rad}{hr}$

$\tan\theta = \frac{400}{x} = 400x^{-1}$

$\sec^2\theta \frac{d\theta}{dt} = -400x^{-2} \frac{dx}{dt}$

when $\theta = \frac{\pi}{6}$

$\tan\frac{\pi}{6} = \frac{400}{x}$

$x = \frac{400}{\tan\frac{\pi}{6}} = 692.8$

$\left(\frac{1}{\cos\frac{\pi}{6}}\right)^2 (-0.25) = \frac{-400}{692.8^2} \frac{dx}{dt}$

$\boxed{400 \frac{ft}{hr} = \frac{dx}{dt}}$