Solutions by Dr. KATIRAIE

Katira

MA 160 Using Derivatives to Learn About Function Behavior - Summary and Examples
Sections 4.2, 4.3, 4.5

I. Summary

1. ***Using the first derivative:***

(a) The sign of the first derivative tells if a function is increasing or decreasing. If f' is positive on an interval then f is increasing on that interval, and if f' is negative on an interval then f is decreasing on that interval.

(b) In order for a function to have a local maximum or minimum at $(c, f(c))$, either $f'(c)$ must be zero or $f'(c)$ must be undefined. Values of x in the domain of f at which this occurs are called ***critical numbers*** of the function.

(c) ***The First Derivative Test:*** Suppose that x = c is a critical value of the function f.

(i) If the first derivative changes sign from positive to negative at (c, f(c)), then the function f changes from increasing to decreasing and has a local maximum at (c, f(c)).

(ii) If the first derivative changes sign from negative to positive at (c, f(c)), then the function f changes from decreasing to increasing and has a local minimum at (c, f(c)).

(iii) If the first derivative does not change sign at (c, f(c)), then f does not have a local extreme value at (c, f(c)).

2. ***Using the second derivative:***

(a) The sign of the second derivative tells if a function is concave up or concave down. If f'' is positive on an interval then f is concave up on that interval, and if f'' is negative on an interval then f is concave down on that interval.

(b) Since the second derivative of a function f is the derivative of f', the sign of the second derivative tells whether f' is increasing or decreasing.

(c) The second derivative can often be used to determine if a function has a local maximum or a local minimum at a critical value.

The Second Derivative Test: If x = c is a critical value of the function f, then

(i) If $f''(c) < 0$, the function f is concave down at (c, f(c)) and has a local maximum at $(c, f(c))$.

(ii) If $f''(c) > 0$, the function f is concave up at (c, f(c)) and has a local minimum at $(c, f(c))$.

(iii) If $f''(c) = 0$, then there is no conclusion and the First Derivative Test should be used.

(d) If the second derivative changes sign (from positive to negative or from negative to positive) at $(x_0, f(x_0))$, then the function f is said to have an ***inflection point*** at $(x_0, f(x_0))$.

(d) In order for a function to have an inflection point at $(x_0, f(x_0))$, either $f''(x_0)$ must be zero or $f''(x_0)$ must be undefined. Points in the domain of f at which this occurs are ***candidates for inflection points*** of the function.

3. Let $f(x)=x^4-4x^3-7$.

(a) Find f' and f''.

$\boxed{f'(x)=4x^3-12x^2}$

$\boxed{f''(x)=12x^2-24x}$

(b) List the critical values of the function.

let $4x^3-12x^2=0$ $\quad 4x^2(x-3)=0$ $\quad 4x^2=0$ $\quad x-3=0$

$\boxed{x=0}$ $\quad \boxed{x=3}$

(c) Determine the intervals on which the function is increasing and those on which it is decreasing and then state whether the function has a relative maximum, a relative minimum, or neither at each critical value listed in part (b).

local min occurs at $x=3$

	$(-\infty,0)$	$x=0$ $(0,3)$	$x=3$ $(3,\infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	↘	↘	↗

(d) Determine the possible points of inflection of the function.

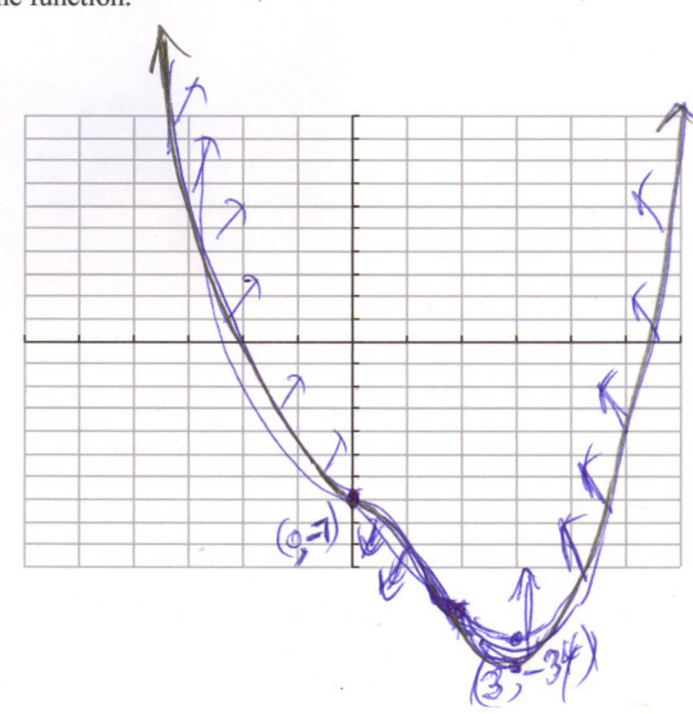
let $f''(x)=0 \Rightarrow 12x^2-24x=0$ $\quad 12x(x-2)=0$ $\quad x=0$, $x=2$

(e) Determine the intervals on which the function is concave up and those on which it is concave down and then state whether the function actually has an inflection point at each point in part (d).

	$(-\infty,0)$	$x=0$ $(0,2)$	$x=2$ $(2,\infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	CU	CD	CU

(f) Determine the end behavior of the function.

(g) Using the information above, sketch a graph of the function, showing any local extreme points and inflection points.



OVER→

3. *End behavior of a function*

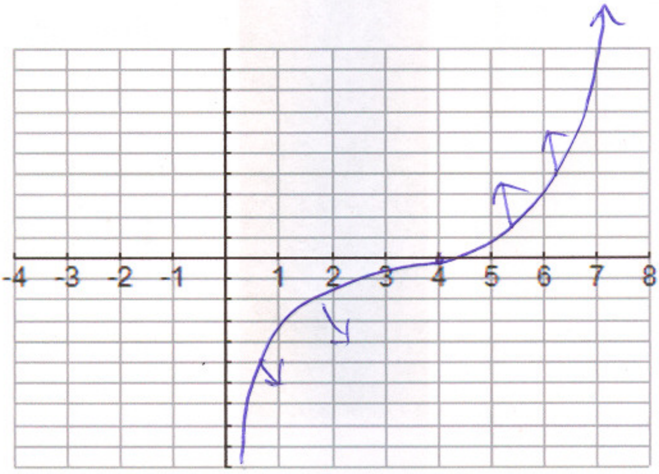
Investigating the ***end behavior of a function*** means determining how the function behaves for x very large in the positive and negative directions, that is, determining $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$. We will only be considering the end behavior of a polynomial function.

For a polynomial function, the end behavior is determined by the behavior of the term of highest degree.

For example, for the polynomial $f(x)=-2x^3-3x^2+7x+2$, the end behavior is determined by the term $2x^3$. As $x\to\infty$, $-2x^3\to-\infty$ and as $x\to-\infty$, $-2x^3\to\infty$.

II. Examples

1. Sketch the graph of a differentiable function which is increasing and concave down on $(-\infty,4)$ and increasing and concave up on $(4,\infty)$.



2. Suppose that a function f is differentiable for all x and $f(3)=5,\ f'(3)=0,\ \textit{and}\ f''(3)=-2$. Only one of the following is true. Which is it?

(a) f has a relative minimum at (3, 0)

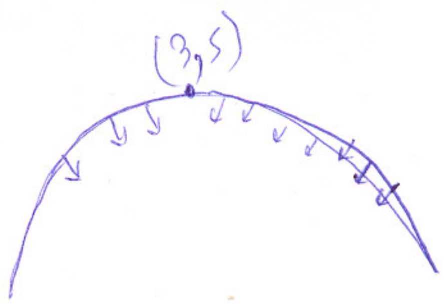
(b) f has a relative maximum at (3, 0)

(c) f has a relative minimum at (3, 5)

(d) f has a relative maximum at (3, 5)

(e) f has a point of inflection at (3, 0)

(f) f has a point of inflection at (3, 5)



4. Sketch the graph of a differentiable function that satisfies all of the given conditions.

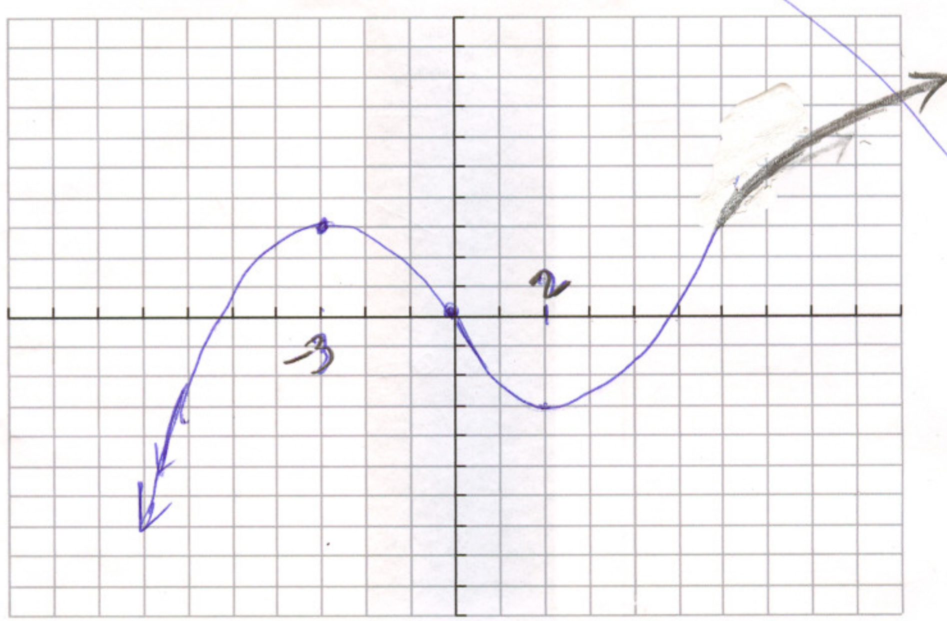
$f'(-3) = f'(2) = 0$

$f'(x) > 0$ *if* $x < -3$ *or* $x > 2$

$f'(x) < 0$ *if* $-3 < x < 2$

$f''(x) < 0$ *if* $x < 0$ *or* $x > 6$

$f''(x) > 0$ *if* $0 < x < 6$



	$(-\infty,-3)$	$x=-3$	$(-3,2)$	$x=2$	$(2,\infty)$
$f'(x)$	+		—		+
$f(x)$	↗		↘		↗

	$(-\infty,0)$	$x=0$	$(0,6)$	$x=6$	$(6,\infty)$
$f''(x)$	—		+		—
$f(x)$	CD		CU		C.D