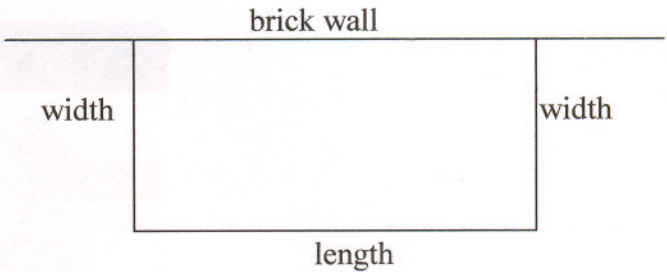


1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a long brick wall. He needs no fencing along the wall. We are going to try to determine the dimensions of the field that will have the largest area.

(a) In order to get a feel for the problem, we'll first fill out a chart showing some possibilities. Keep in mind that the total amount of fencing is 2400 ft.



Width	Length	Area of field
100 ft	2200 ft	220000 ft ²
200 ft	2000 ft	400000 ft ²
500 ft	1400 ft	700000 ft ²
900 ft	600 ft	540000 ft ²

Note that different widths and lengths result in different areas.

(b) Now suppose that the width is w . If the length is l , write an equation relating w and l . You will have to use the fact that the total amount of fencing is 2400 ft.

$$2w + l = 2400$$

(c) Solve the equation for l in part (b) for l .

$$l = 2400 - 2w \text{ feet}$$

(d) Since the area of a rectangle is the product of width times length, $A = lw$. Use your answer to part (c) to rewrite this formula for area in terms of w only.

$$A = w(2400 - 2w) = 2400w - 2w^2 \text{ (ft}^2\text{)}$$

(e) The width of a field must be a positive number, so we must have $w > 0$. The length also must be positive. Use your expression for l to get another restriction on the value of w . These two inequalities give the domain of the area function.

$$0 < w < 1200$$

$$y' = 2400 - 2x^2$$

(f) Use the techniques of Calculus to find the dimensions of the field (the values of the width and the length) that will result in a maximum value of the function A on the interval determined by your answer to part (e). Confirm that your answer gives a maximum and not a minimum by using the First or Second Derivative Test. What is the maximum value of A ?

$A'' = -4$ Always concave down.

$$\frac{dA}{dw} = 2400 - 4w \text{ let it equal to zero } \Rightarrow 2400 - 4w = 0$$

$$2400 = 4w \Rightarrow w = \frac{2400}{4} = 600 \text{ feet}$$

$$\therefore l = 2400 - 2w = 2400 - 2(600) = 1200 \text{ feet}$$

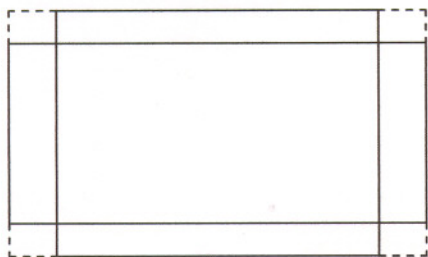
$$\begin{aligned} x_{\min} &= 10 \\ x_{\max} &= 1500 \\ y_{\min} &= -10 \\ y_{\max} &= 80000 \end{aligned}$$

$$A_{\max} = 600 \times 1200 = 720000 \text{ ft}^2$$

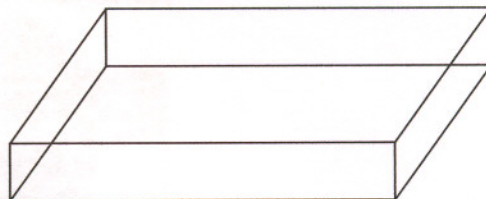
2. You are going to make a box out of a piece of cardboard that is 10 in. by 14 in. by cutting out squares from each corner, turning up the sides and taping the sides to make a box. You want the box to hold as much as possible, so you want the volume of the box to be a maximum. What size square should be cut out of each corner to maximize the volume of the box?

Here's a picture to show what's happening:

Flat piece of cardboard with corners cut out



Cardboard folded up to make a box



- (a) Let x be the length of each side of the square that is cut out from each corner. This will become the height of the box.

Write expressions for the length and the width of the box.

$$L = 14 - 2x \quad W = 10 - 2x$$

- (b) The volume of a box is (length)(width)(height). Using the dimensions you found above, write an expression for the volume $V(x)$ of the box as a function of x .

$$V = x(14 - 2x)(10 - 2x) \\ = (14x - 2x^2)(10 - 2x) = 140x - 28x^2 - 20x^2 + 4x^3 = 4x^3 - 48x^2 + 140x$$

- (c) Since each dimension of the box must be positive, use this to write the domain of $V(x)$.

$$0 < x < 5 \text{ inches}$$

- (d) Use the techniques of Calculus to find the dimensions of the box that will result in a maximum value of the volume function $V(x)$ on the interval determined by your answer to part (c). Confirm that your answer gives a maximum and not a minimum by using the First or Second Derivative Test. What is the maximum volume?

$$V(x) = 4x^3 - 48x^2 + 140x$$

$$V'' = 24x - 96$$

$$\frac{dV}{dx} = 12x^2 - 96x + 140 \Rightarrow \text{let } \frac{dV}{dx} = 0 \quad a=12 \quad b=-96 \quad c=140$$

$$x = \frac{96 \pm \sqrt{(-96)^2 - 4(12)(140)}}{2(12)} = \frac{96 \pm 50}{24} = 6.08; \quad V''(1.92) = -49.92 < 0 \text{ C.D.}$$

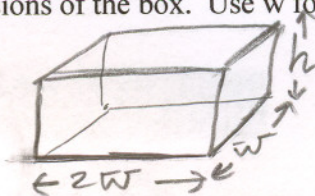
$$\hookrightarrow \frac{96 - 50}{24} = 1.92 \text{ inches}$$

$$\text{Max volume} = 120.16 \text{ inch}^3$$

3. A rectangular box open at the top is to be constructed so that the volume is 10 cubic meters and the base has length equal to twice its width. Our goal is to find the dimensions of the box that will minimize the surface area of the box.

To help you in solving this problem, follow these steps:

- (a) Draw a picture and label the dimensions of the box. Use w for the width and h for the height.



- (b) Write an expression in terms of h and w for the surface area S of the box. Keep in mind that the box has five surfaces --- the bottom plus four sides.

$$\begin{aligned} \text{Surface Areas} &= \text{Bottom} + \text{Right side} + \text{left side} + \text{Front side} + \text{Back side} \\ &= 2w^2 + wh + wh + 2wh + 2wh \\ &= 2w^2 + 6wh \end{aligned}$$

- (c) Use the fact that the volume of the box is 10 m^3 to relate h and w and then solve for h in terms of w .

$$V = w(2w)(h) \implies 10 = 2w^2h \implies h = \frac{5}{w^2}$$

- (d) Substitute the expression for h that you found in (c) into the expression for S in (b).

$$\text{Surface Area} = 2w^2 + 6w\left(\frac{5}{w^2}\right) = 2w^2 + 30w^{-1}$$

- (e) You should now have an expression for S in terms of w only. Use calculus to find the critical value of S . Then use either the First or Second Derivative Test to determine that S actually has a minimum as opposed to a maximum at the critical value.

$$(\text{Surface Area})' = 4w - 30w^{-2}$$

$$(\text{SA})'' = 4 + 60w^{-3}$$

$$\text{SA}''(1.957) = + \text{C.U.}$$

$$4w - \frac{30}{w^2} = 0$$

$$4w = \frac{30}{w^2} \implies 4w^3 = 30$$

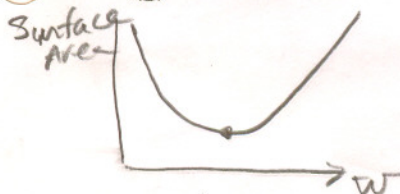
$$w = \sqrt[3]{\frac{30}{4}} = 1.957 \text{ meters}$$

- (f) What are the dimensions of the box that will result in a minimum surface area?

$$\text{If } w = 1.957 \text{ meter, Length} = 2w = 2 * 1.957 = 3.914 \text{ meter}$$

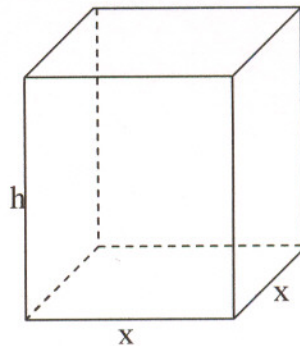
$$h = \frac{5}{w^2} = \frac{5}{(1.957)^2} = 1.306 \text{ meter}$$

- (g) What is the minimum surface area?



$$\text{When } w = 1.957 \text{ meter} \implies \text{Surface Area} = 22.989 \text{ m}^2$$

4. You've gotten so good at constructing boxes that you have agreed to construct a bunch of boxes for your little sister and brother and their friends to hold some of their small toys. Your brother and sister insist that the base of the box be square, and that the volume of the box be 6 cubic feet. (They're both a little strange.) You are going to make the bottom of the box out of heavy cardboard costing 10 cents per square foot and the sides out of lighter cardboard costing 3 cents per square foot. The box will not have a top. Naturally, you want to spend as little as possible on doing this good deed. Using x as the side of the square in the base and h as the height as in the box shown,



- (a) Write an expression for the cost of the cardboard for the box.

$$\begin{aligned} \text{Cost} &= 10x^2 + 3(4xh) \\ &= 10x^2 + 12xh \quad (\text{cents}) \end{aligned}$$

- (b) Use the fact that the volume of the box is to be 6 cubic feet to write an equation relating x and h .

$$V = x^2h \Rightarrow x^2h = 6 \Rightarrow h = \frac{6}{x^2}$$

- (c) Solve the equation in part (b) for h .

$$h = \frac{6}{x^2}$$

- (d) Substitute the expression that you got for h in part (c) into the cost expression that you found in part (a) and simplify your answer.

$$\text{Cost} = 10x^2 + 12x\left(\frac{6}{x^2}\right) = 10x^2 + 72x^{-1}$$

- (e) You should now have an expression for cost that is a function of x only. Use the techniques of Calculus to find the value of x (to the nearest tenth of a foot) that will result in a minimum cost.

$$\frac{d(\text{Cost})}{dx} = 20x - 72x^{-2} \Rightarrow 20x - \frac{72}{x^2} = 0 \Rightarrow 20x^3 = 72$$

$$\text{Cost}'' = 20 + 144x^{-3} > 0 \quad x^3 = \frac{72}{20} \Rightarrow x = \sqrt[3]{\frac{72}{20}} = 1.53 \text{ feet} \approx 1.5 \text{ foot}$$

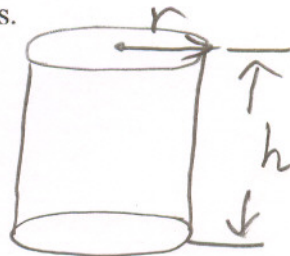
- (f) What are the dimensions of the box of minimum cost and what is the minimum cost for each box?

$$x = 1.53 \text{ feet} \quad h = \frac{6}{x^2} = \frac{6}{1.5^2} = 2.7 \text{ feet}$$

$$\begin{aligned} \text{Cost} &= 10x^2 + 72x^{-1} \\ &= 10(1.5)^2 + 72(1.5)^{-1} = 70.5 \text{ cents} = \boxed{71 \text{ cents}} \end{aligned}$$

5. We are now going to consider the problem of determining the dimensions of a cylindrical can with total minimum surface area which has a volume of 280 cubic inches.

(a) Draw a picture of the can and label the radius r and the height h .



(b) The surface area of the can consists of the sum of the area of the top and bottom plus the area of the side of the can (called the *lateral surface area*).

The top and bottom are circles so each of these has an area of πr^2 . The lateral surface area of a cylinder is $2\pi rh$. Use this information to write a formula for the total surface area of the can.

$$\begin{aligned} \text{Surface Area} &= \text{top} + \text{Bottom} + \text{lateral Surface} \\ &= \pi r^2 + \pi r^2 + 2\pi r h \end{aligned}$$

(c) The volume of a cylinder is $V = \pi r^2 h$. Use this formula and the fact that the volume of the can is 280 cubic inches to write an equation relating r and h and then solve this equation for h .

$$S.A. = 2\pi r^2 + 2\pi r h$$

$$280 = \pi r^2 h \Rightarrow h = \frac{280}{\pi r^2}$$

(d) Substitute the expression for h that you found in part (c) into the expression for surface area in part (b).

$$\text{Surface Area} = 2\pi r^2 + 2\pi r \left(\frac{280}{\pi r^2} \right)$$

$$S.A. = 2\pi r^2 + 560r^{-1}$$

(e) You should now have an expression for S in terms of r only. Use calculus to find the critical number of S . Then use either the First or Second Derivative Test to determine that S actually has a minimum as opposed to a maximum at the critical number.

$$(SA)' = 4\pi r - 560r^{-2}$$

$$4\pi r - \frac{560}{r^2} = 0 \Rightarrow 4\pi r^3 = 560 \Rightarrow r = \sqrt[3]{\frac{560}{4\pi}} = 3.545 \text{ inches}$$

$$(SA)'' = 4\pi + 1120r^{-3} \Rightarrow SA''(3.545) = 4\pi + 1120(3.545)^{-3} = 37.71 \text{ c.u.}$$

(f) What are the dimensions of the can that will result in a minimum surface area?

$$r = 3.545 \text{ inches and } h = \frac{280}{\pi r^2} = \frac{280}{\pi(3.545)^2} = 7.09 \text{ inches}$$

(g) What is the minimum surface area?

$$SA = 2\pi r^2 + 560r^{-1} = 2\pi(3.545)^2 + 560(3.545)^{-1} = 236.93 \text{ inch}^2$$