Group Work 1, Section 3.5
Unbroken Chain

For each of the following functions of x, write the equation for the derivative function. This will go a lot more smoothly if you remember the Sum, Product, Quotient, and Chain Rules... especially the Chain Rule! Please do us both a favor and don't simplify the answers.

1. $f(x) = \sin 3x$ $\qquad f'(x) = 3\cos(3X)$

   $f'(x) = \cos(3x) \cdot 3$

2. $g(x) = (\sin 3x)^3$ $\qquad g'(x) = 3(\sin(3x))^2(\cos 3x)(3)$

   $= 9\sin^2(3x)\cos(3x)$

3. $h(x) = (\sin 3x)^3 + 5x$ $\qquad h'(x) = 9\sin^2(3x)\cos(3x) + 5$

4. $j(x) = \left[(\sin 3x)^3 + 5x\right]^2$ $\qquad j'(x) =$

   $j'(x) = 2\left[(\sin(3x))^3 + 5x\right]^1\left[9\sin^2(3x)\cos(3x) + 5\right]$

5. $k(x) = x + \dfrac{1}{x} = X + X^{-1}$ $\qquad k'(x) = 1 - 1X^{-2} = 1 - \dfrac{1}{X^2}$

6. $l(x) = \sqrt{x + \dfrac{1}{x}} = \left(X + \dfrac{1}{X}\right)^{\frac{1}{2}}$ $\qquad l'(x) = \frac{1}{2}\left(X + \frac{1}{X}\right)^{-\frac{1}{2}}\left(1 - \frac{1}{X^2}\right)$

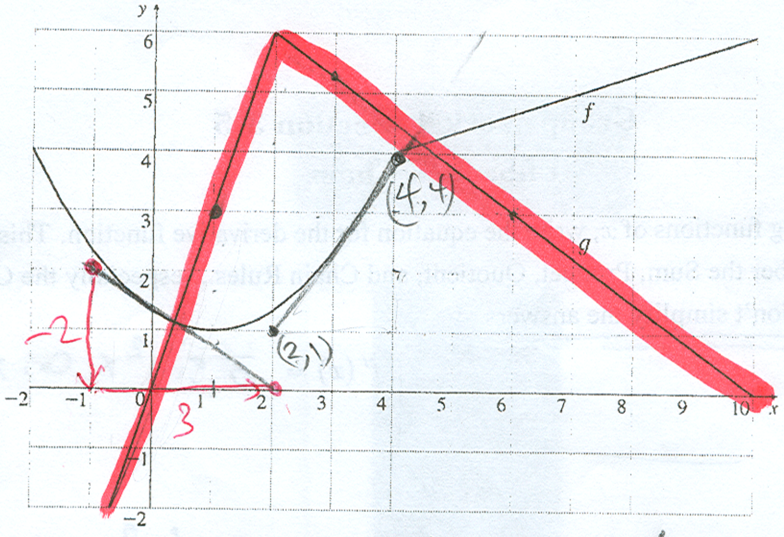
   $= \dfrac{1\left(1 - \frac{1}{X^2}\right)}{2\sqrt{1 + \frac{1}{X}}}$

7. $m(x) = \left(\sqrt{x + \dfrac{1}{x}}\right)\left[(\sin 3x)^3 + 5x\right]^2$ $\qquad m'(x) =$

$$m'(x) = \frac{1\left(1 - \frac{1}{x^2}\right)}{2\sqrt{1 + \frac{1}{x}}}\left[(\sin 3x)^3 + 5x\right]^2 + 2\left[(\sin 3x)^3 + 5x\right]\left[9\sin^2(3x)\cos(3x) + 5\right]\left[\sqrt{X + \frac{1}{X}}\right]$$

Group Work 2, Section 3.5
Chain Rule Without Formulas

Consider the functions f and g given by the following graph:



Define $h = f \circ g. = f(g(x))$ $\quad h'(x) = f'(g(x))\cdot g'(x)$

1. Compute $h'(1)$.

$$h'(1) = f'(g(1))\cdot g'(1) = f'(3)\, g'(1) = \left(\frac{\overset{\text{Rise}}{3}}{\underset{\text{Run}}{2}}\right)\left(\frac{3}{1}\right) = \frac{9}{2}$$

2. Compute $h'(0)$. $\quad h'(0) = f'(g(0))\, g'(0)$

$$= f'(0)\; g'(0) = \left(\frac{-2}{3}\right)\left(\frac{3}{1}\right) = -2$$

3. Does $h'(2)$ exist? $\quad h'(2) = f'(g(2))\, g'(2)$

$$= f'(6)\; g'(2)$$

But $g'(2)$ DNE B/c of the sharp edge

So, $h'(2)$ DNE !!

Worksheet for Section 3.6
Circles and Implicit Differentiation

Solve the following by Implicit Differentiation method.
Do Not Use Calculator

1. Consider the circle $x^2 + y^2 = 1$.
   a) At what point(s) is the slope of the tangent line equal to 1?

$2x + 2yy' = 0 \Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y} \Rightarrow m = \frac{-x}{y}$

We want $\frac{-x}{y} = 1 \Rightarrow \boxed{y = -x}$ plug it in $x^2 + (-x)^2 = 1$ $2x^2 = 1$

$x^2 = 1/2 \Rightarrow \boxed{x = \pm\sqrt{1/2};\ y = \mp\sqrt{}}$

   b) At what point(s) is the slope of the tangent line equal to -1?

We want $\frac{-x}{y} = -1 \Rightarrow \boxed{y = x}$ plug it in we get $x^2 + x^2 = 1$

$2x^2 = 1 \Rightarrow x^2 = 1/2 \Rightarrow \boxed{x = \pm\sqrt{1/2} \text{ and } y = \pm\sqrt{1/2}}$

   c) At what point(s) is the slope of the tangent line equal to 0?

We want $\frac{-x}{y} = 0 \Rightarrow 0 = x \Rightarrow 0^2 + y^2 = 1 \Rightarrow y = \pm 1$

$\boxed{(0,1) \text{ and } (0,-1)}$

2. Consider the circle $x^4 + y^4 = 1$.
   a) At what point(s) is the slope of the tangent line equal to 1?

$4x^3 + 4y^3y' = 0 \Rightarrow y' = \frac{-x^3}{y^3} \Rightarrow 1 = \frac{-x^3}{y^3} \Rightarrow y^3 = -x^3$ $y = -x$

$x^4 + (-x)^4 = 1 \Rightarrow 2x^4 = 1 \Rightarrow x^4 = 1/2$ $x = \pm\sqrt[4]{\frac{1}{2}}$ $\left(\pm\sqrt[4]{\frac{1}{2}}, \mp\sqrt[4]{\frac{1}{2}}\right.$

   b) At what point(s) is the slope of the tangent line equal to -1?

$\frac{-x^3}{y^3} = -1 \Rightarrow y^3 = x^3 \Rightarrow \boxed{y = x}$

$x^4 + (x)^4 = 1$ $2x^4 = 1 \Rightarrow x^4 = 1/2 \Rightarrow \left(\pm\sqrt[4]{\frac{1}{2}}, \pm\sqrt[4]{\frac{1}{2}}\right)$

   c) At what point(s) is the slope of the tangent line equal to 0?

$\frac{-x^3}{y^3} = 0 \Rightarrow x^3 = 0 \Rightarrow x = 0$ $0^4 + y^4 = 1$ $y^4 = 1$

$(0, 1)$ $(0, -1)$ $y = \pm$

Group Work 1, Section 3.7
Logarithmic Differentiation

1. Let $f(x) = (x+1)^x$. We want to compute $f'(x)$.

(a) Use logarithmic differentiation to compute $f'(x)$.

let $y = (x+1)^x$

$\ln y = x \ln(x+1)$

$\Rightarrow \frac{1}{y} y' = 1 \ln(x+1) + x \frac{1}{x+1}$

$y' = y\left[\ln(x+1) + \frac{x}{x+1}\right]$

$= (x+1)^x \left[\ln(x+1) + \frac{x}{x+1}\right]$

(b) Let $g(x) = e^{x \ln(x+1)}$. What is the relationship between $f(x)$ and $g(x)$?

$g(x) = e^{\ln(x+1)^x} = (x+1)^x$ then $f(x)$ and $g(x)$ are the same

(c) Compute $g'(x)$. What techniques are you using in this case?

$g'(x) = e^{x\ln(x+1)} \cdot \left[1 \ln(x+1) + x \frac{1}{x+1}\right] = e^{\ln(x+1)^x}\left[\ln(x+1) + x\left(\frac{1}{x+1}\right)\right]$

$= (x+1)^x \left[\ln(x+1) + \frac{x}{x+1}\right]$

2. Let $f(x) = (x^2+1)^{\sin x}$. Compute $f'(x)$.

$y = (x^2+1)^{\sin x} \Rightarrow \ln y = \ln (x^2+1)^{\sin x} \Rightarrow \ln y = \sin x \cdot \ln(x^2+1)$

$\frac{1}{y} y' = \cos x \ln(x^2+1) + \sin x \frac{1}{x^2+1} \cdot 2x \Rightarrow y' = y\left[\cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right]$

$y' = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]$

3. Let $y = f(x)$ be implicitly defined by $x^{\sin y} = y^{\cos x}$. Compute y' in terms of x and y. (*Hint:* Can logarithms help you?)

$\ln x^{\sin y} = \ln y^{\cos x} \Rightarrow \sin y \ln x = \cos x \ln y$

$\cos y \; y' \cdot \ln x + \sin y \frac{1}{x} = -\sin x \ln y + \cos x \frac{1}{y} y'$

$y'\left[\cos y \ln x - \frac{\cos x}{y}\right] = -\sin x \ln y - \sin y \cdot \frac{1}{x}$

$y' = \dfrac{-\sin x \ln y - \frac{\sin y}{x}}{\cos y \ln x - \frac{\cos x}{y}}$

Group Work 3, Section 3.7

e^2 as a Limit

1. If $f(x) = \ln x$, what is $f'\left(\frac{1}{2}\right)$?

$$f'(x) = \frac{1}{x} \Longrightarrow f'\left(\tfrac{1}{2}\right) = \frac{1}{1/2} = 2$$

2. Use the definition of $f'(x)$ to find a formula involving limits at 0 for $f'\left(\frac{1}{2}\right)$.

$$f'\left(\tfrac{1}{2}\right) = \lim_{h\to 0} \frac{\ln\left(\frac{1}{2}+h\right) - \ln\left(\frac{1}{2}\right)}{h} =$$ By Calculator this approaches 2 as $h \to 0$

Recall $\boxed{\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)}$

3. Show that the formula in Problem 2 can be written as $\lim_{h\to 0} \ln(1+2h)^{1/h}$.

$$\lim_{h\to 0} \frac{\ln\left(\frac{1}{2}+h\right) - \ln\left(\frac{1}{2}\right)}{h} \Rightarrow \frac{\ln\left(\frac{\frac{1}{2}+h}{\frac{1}{2}}\right)}{h} \Rightarrow \frac{\ln\left(\frac{1+2h}{1}\right)}{h} \Rightarrow \frac{1}{h}\cdot \ln(1+2h)$$

$$= \boxed{\lim_{h\to 0} \ln(1+2h)^{1/h}}$$

4. Using Problems 1 and 2, show that $e^2 = \lim_{h\to 0}(1+2h)^{1/h}$.

Recall $f'(2) = \lim_{h\to 0} \ln(1+2h)^{\frac{1}{h}} = 2$ Raise Both sides by e

$$e^{\ln \lim_{h\to 0} (1+2h)^{\frac{1}{h}}} = e^2 \Longrightarrow \boxed{\lim_{h\to 0} (1+2h)^{\frac{1}{h}} = e^2}$$

5. Rewrite the equation in Problem 4 to obtain $e^2 = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$.

let $\frac{1}{h} = n$

as $h \to 0$

$n \to \infty$

$$\Rightarrow e^2 = \lim_{h\to 0} (1+2h)^{\frac{1}{h}} = \lim_{n\to\infty}\left(1+2\cdot\frac{1}{n}\right)^n = \boxed{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n}$$

6. What do you think is the appropriate formula for e^3? For e^5? For e^m, where m is any integer?

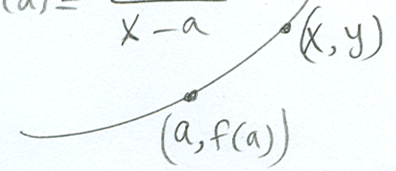
$$e^3 = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n$$

$$e^5 = \lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$$

$$\boxed{e^m = \lim_{n\to\infty}\left(1+\frac{m}{n}\right)^n}$$

$\boxed{L = y = f(a) + f'(a)(x-a)}$

$f'(a) = \dfrac{y - f(a)}{x - a}$



Group Work 1, Section 3.8
Four Variations on a Theme

Consider the following four functions:

$f(x) = e^{x-1} - 1 \qquad g(x) = x^4 - 3x + 2 \qquad h(x) = \ln x \qquad j(x) = \frac{2}{\pi}\sin\left(\frac{\pi}{2}(x-1)\right)$

1. Find the linearizations of f, g, h, and j at $a = 1$.

$f'(x) = e^{x-1}$

$f'(1) = e^{1-1} = e^0 = 1$

$L = f(1) + f'(1)(x-1)$

$f(1) = e^{1-1} - 1 = e^0 - 1 = 0$

Thus $L = 0 + 1(x-1)$

$\boxed{L = x - 1}$

$g'(x) = 4x^3 - 3$

$g'(1) = 4(1) - 3 = 1$

$L = g(1) + g'(1)(x-1)$

$g(1) = 1 - 3 + 2 = 0$

Thus $L = 0 + 1(x-1)$

$\boxed{L = x - 1}$

$h'(x) = \frac{1}{x}$

$h'(1) = \frac{1}{1} = 1$

$L = h(1) + h'(1)(x-1)$

$h(1) = \ln 1 = 0$

Thus

$L = 0 + 1(x-1)$

$\boxed{L = x - 1}$

$j'(x) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}(x-1)\right)\cdot\frac{\pi}{2}$

$j'(1) = \cos\left(\frac{\pi}{2}(1-1)\right)$

$= \cos\left(\frac{\pi}{2}(0)\right) = \cos 0 =$

$L = j(1) + j'(1)(x-1)$

$= 0 + 1(x-1)$

$\boxed{L = 1(x-1)}$

2. Compute the values of each of these functions at $x = 1.1$ and the values of their linearizations. For which function is the approximation best? For which is it worst? Why?

$L = x - 1 = 1.1 - 1 = 0.1$

$e^{1.1-1} - 1 = 0.105$

So, these values are very close

Good approximation

$L = 1.1 - 1 = 0.1$

$(1.1)^4 - 3(1.1) + 2 = 0.164$

These values are not very close to eachother

Bad Approximation

$L = 1.1 - 1 = 0.1$

$\ln(1.1) = 0.095$

≈ 0.1

these values are close !!

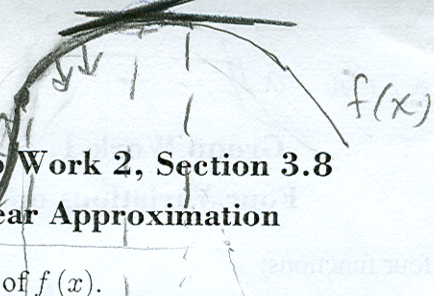
Good Approximation

$L = 1.1 - 1 = 0.1$

$\frac{2}{\pi}\sin\left(\frac{\pi}{2}(1.1-1)\right)$

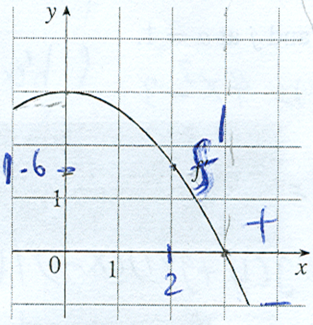
$= 0.0996$

Good Approximation



Group Work 2, Section 3.8
Linear Approximation

Consider this graph of $f'(x)$, the *derivative* of $f(x)$.



Recall $L = f(a) + f'(a)(x-a)$

1. Suppose that $f(2) = 4$. Approximate $f(1.98)$ and $f(2.02)$ as best you can. Don't just guess. Show your work.

$(2,4)$

$L = f(2) + f'(2)(x-2)$

$= 4 + 1.6(x-2)$

$= 4 + 1.6x - 3.2$

$\boxed{L = 0.8 + 1.6x}$

$f(1.98) = 0.8 + 1.6(1.98) = 3.968$

$f(2.02) = 0.8 + 1.6(2.02) = 4.032$

2. Determine whether your approximations were overestimates or underestimates.

overestimates

3. Suppose you also know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer.

$(3,7)$

$L = f(3) + f'(3)(x-3)$

$= f(3) + 0(x-3)$

$L = f(3)$

$L = 7$; the Linear approximation is a flat horizontal line So it may still be a Good approximation