Group Work 3, Section 4.3
Graphing with the Derivative (Form A)

This exercise is designed to illustrate how numerical information from a function and its derivatives can be used to get a very good sense of how the function looks. While it is a good idea to use your graphing calculator to check your final answers, it would be missing the point to use it earlier.

Consider the function

$$f(x) = \frac{-4x-3}{x^2-x+4}$$

$f'(x) = \frac{-4(x^2-x+4)-(2x-1)\ldots}{(x^2-x+4)^2}$

1. Where are the zeros (roots) of this function?

$f(x)=0$ when $-4x-3=0 \Rightarrow -4x=3$

$\boxed{x = -\frac{3}{4}}$

2. On what intervals is this function increasing? On what intervals is it decreasing?

$x > 1.55$
$x < -3.05$

$-3.05 < x < 1.55$

3. Where are the local maxima and minima?

Max at $x=-3.05$
$y = 0.563$

$x = 1.55$
$y = -1.896$

4. It is a fact that $f''(x)$ simplifies to $\frac{-8x^3-18x^2+114x-14}{(x^2-x+4)^3}$. Where is f concave up? Where is f concave down?

$f''(x) > 0$ concave up $2.73 > x > 0.125$ and $x < -5.11$

$f''(x) < 0$ concave down $-5.11 < x < 0.125$

5. Where are the inflection points?

point where the 2nd derivative changes sign.

$x = 2.73$ & $x = 0.125$ and $x = -5.11$

6. Does this graph have any vertical asymptotes? If so, what are they? If not, why not?

None, since $x^2 - x + 4 \neq 0$

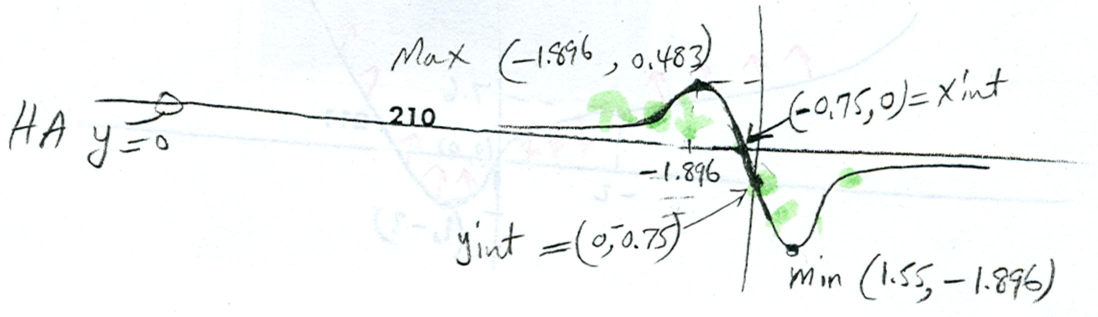
$x = \frac{1 \pm \sqrt{(1)^2 - 4(1)(4)}}{2(1)}$ // No Re[al] sl[n]

7. What appears to happen to $f(x)$ when x gets very large? What appears to happen when x gets very large and negative? What appears to happen when $x = 0$?

$y$int $= -0.75$

$\boxed{y \to 0 \text{ H.A.}}$

8. Using this information, sketch a graph of this function on a separate piece of paper.



Group Work 3, Section 4.3

Graphing with the Derivative (Form B)

This exercise is designed to illustrate how numerical information from a function and its derivatives can be used to get a very good sense of how the function looks. While it is a good idea to use your graphing calculator to check your final answers, it would be missing the point to use it earlier.

Consider the function

$$f(x) = x^{1/3}(x-4)$$

1. Where are the zeros (roots) of f?

$x = 0$ and $x = 4$

2. Does f' have any points x where $f'(x) = 0$? Where $f'(x)$ is not defined?

yes at $\boxed{x=1}$ $\boxed{y=-3}$

yes at $x=0$ $y=0$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x-4) + 1(x)^{\frac{1}{3}} = \frac{x-4}{3x^{2/3}} + 1x^{\frac{1}{3}} = \frac{x-4+3x}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}$$

3. On what intervals is f increasing? On what intervals is it decreasing?

$(1, \infty)$ | $(-\infty, 1)$

4. Where are the local maxima and minima of f?

Local Max None | Local Min at $x = 1$, $y = -3$

5. Does f'' have any points x where $f''(x) = 0$? Where $f''(x)$ is not defined?

$$f''(x) = \frac{4x+8}{9x^{5/3}}$$

at $x = -2$, $y = 7.6$

at $x = 0$, $y = 0$

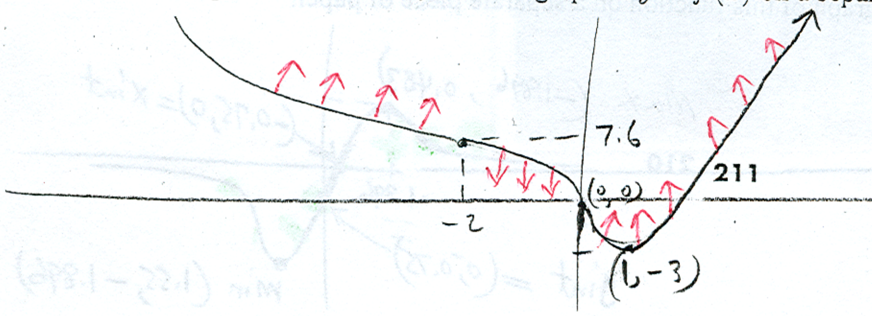
6. Where is f concave up? Where is it concave down?

$(-\infty, -2)$, $(0, \infty)$ | $(-2, 0)$

7. Where are the inflection points of f?

$x = -2$, $y = 7.6$ and $x = 0$, $y = 0$

8. Using this information, sketch a graph of $y = f(x)$ on a separate piece of paper.



Group Work 2, Section 4.4
To Infinity and Beyond

Consider the function $f(x) = \dfrac{2x^4 - 7x^2 - 4}{(x^2-1)^2}$.

1. Identify the values of x where $f(x) = 0$.

$2x^4 - 7x^2 - 4 = 0$

$(2x^2+1)(x^2-4) = 0$ $\boxed{x = \pm 2}$

2. Describe the asymptotic behavior of f as $x \to \infty$ and as $x \to -\infty$.

as $x \to \infty$ $y \to 2$ $\Rightarrow y = 2$ is H A

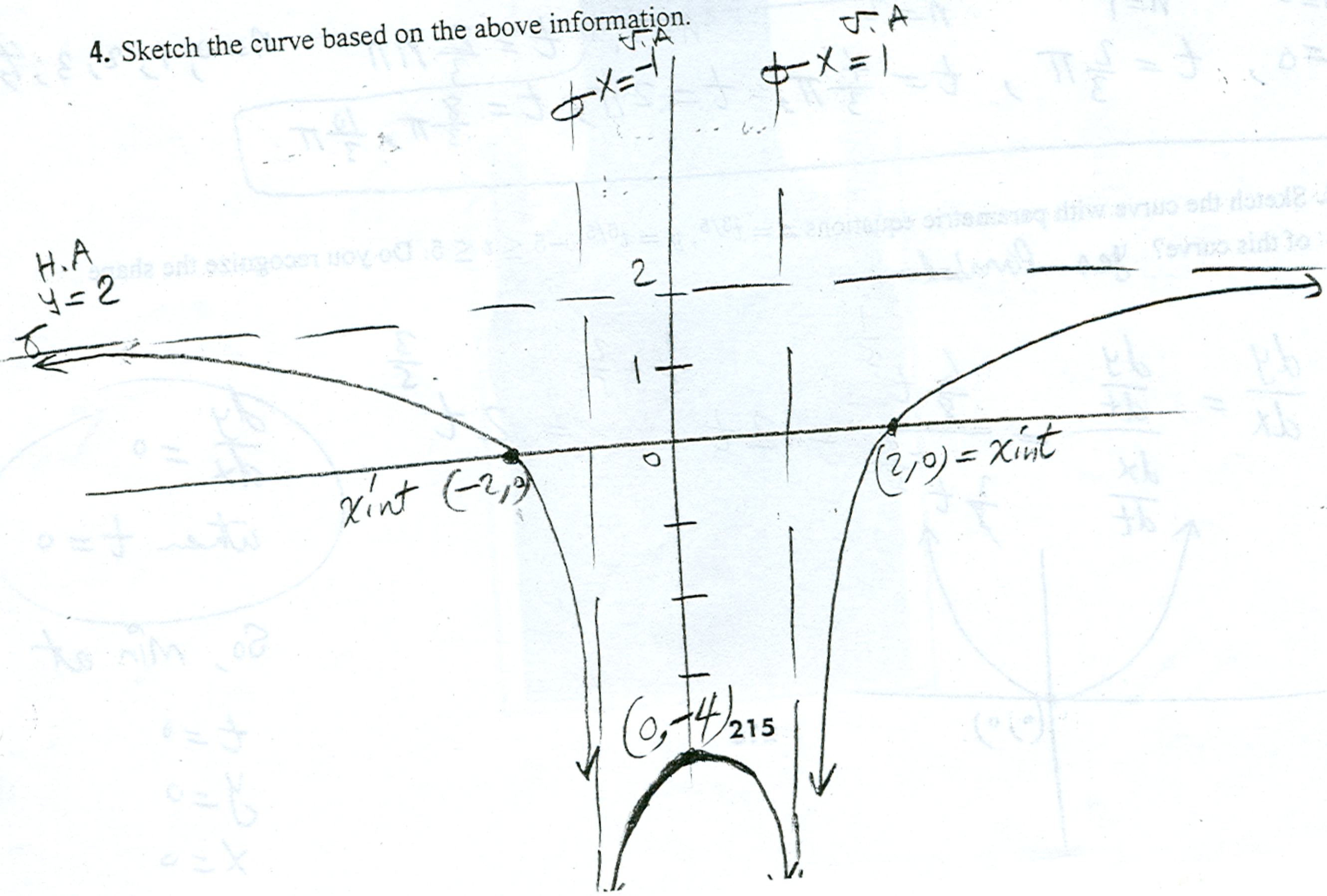
as $x \to -\infty$ $y \to 2$

3. Identify any vertical asymptotes.

$x^2 - 1 = 0$

$x^2 = 1$ $\longrightarrow$ $\boxed{x = \pm 1}$

4. Sketch the curve based on the above information.



Group Work 3, Section 4.4
Puzzling Parametric Curves

1. Using your graphing calculator or computer, graph the curve with parametric equations $x = \sin t$, $y = \cos\left(\frac{3}{2}t\right)$, $0 \le t \le 4\pi$.

(a) By zooming and tracing, find the approximate coordinates of all points at which the tangent line is horizontal.

$t=0 \quad x=0,\ y=1$

$t=2.1 \quad x=0.866,\ y=-1$

$t=4.19 \quad x=-0.866 \quad y=1$

$t=6.3 \quad x=0 \quad ,\ y=-1$

$t=8.38 \quad x=.9866 \quad y=1$

$t=10.47 \quad x=-0.866 \quad y=-1$

(b) By first computing dy/dx, find the exact coordinates of all points at which the tangent line is horizontal.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin\left(\frac{3}{2}t\right)\left(\frac{3}{2}\right)}{\cos(t)} = 0 \Rightarrow \sin\left(\frac{3}{2}t\right) = 0$$

$$\frac{3}{2}t = n\pi \quad n = 0, 1, 2, \ldots$$

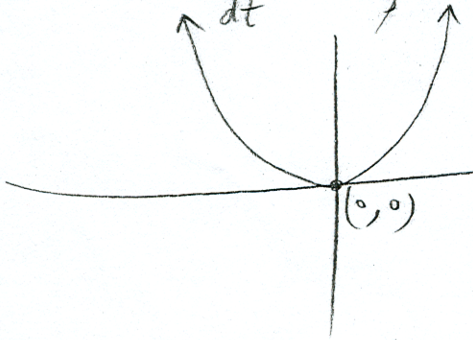
$$t = \frac{2}{3}n\pi \quad n = 0, 1, 2, 3, 4, 5$$

$n=0 \quad n=1 \quad n=2 \quad n=3$

$$\boxed{t=0,\ t=\frac{2}{3}\pi,\ t=\frac{4}{3}\pi,\ t=2\pi,\ t=\frac{8}{3}\pi,\ \frac{10}{3}\pi}$$

2. Sketch the curve with parametric equations $x = t^{3/5}$, $y = t^{6/5}$, $-5 \le t \le 5$. Do you recognize the shape of this curve? yes Parabola

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{6}{5}t^{\frac{1}{5}}}{\frac{3}{5}t^{-\frac{2}{5}}} = 2t^{\frac{1}{5}+\frac{2}{5}} = 2t^{\frac{3}{5}}$$



$\frac{dy}{dx} = 0$ when $t = 0$

So, min at

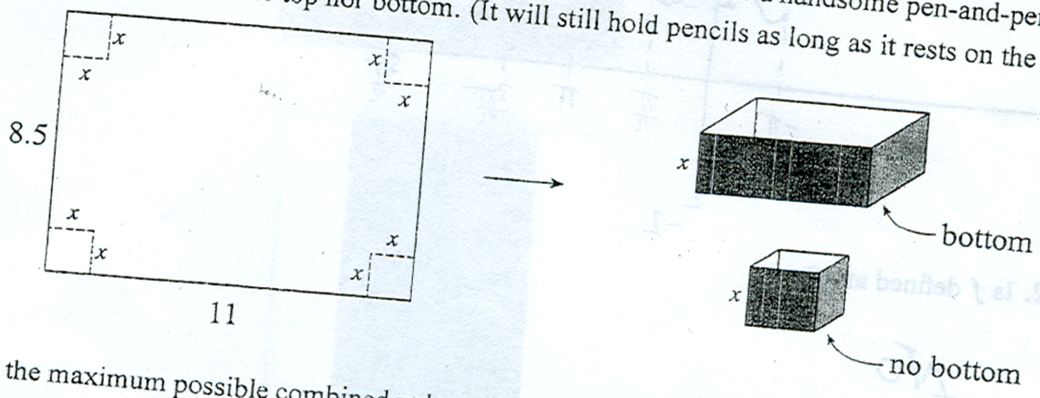
$t=0$

$y=0$

$x=0$

Group Work 2, Section 4.6
The Waste-Free Box

There is a traditional problem (your text has an example in Exercise 8 on page 313) that goes like this: want to make an open-topped box from an 8.5 × 11 inch sheet of paper by cutting congruent squares f the corners and folding up the sides. What is the maximum possible volume of such a box?" What n people never think about is the fate of those four squares of paper. They don't have to be wasted. By tap them together, and putting the resultant structure on a desk, one can make a handsome pen-and-pencil hol which will be a box with neither top nor bottom. (It will still hold pencils as long as it rests on the desk.)



1. What is the maximum possible combined volume of an open-topped box plus a handsome pen-and-pencil holder that can be made by cutting four squares from an 8.5 × 11 inch sheet of paper?

$V = (11-2x)(8.5-2x)(x) + x^3$

$V = (11x-2x^2)(8.5-2x) + x^3$

$V' = (11-4x)(8.5-2x) - 2(11x-2x^2) + 3x^2 = 0$

$93.5 - 22x - 34x + 8x^2 - 22x + 4x^2 + 3x^2 = 0$

$15x^2 - 78x + 93.5 = 0$

$x = 1.87$ inches ↑ max $\quad x = 3.33$ ↑ min

(at $x = 1.87$ inches, $V_{max} \cong 71.$ inch 74

(at $x = 3.33$, $V_{min} = 63.52$ in

2. Describe the open-topped box that results from this maximal case. Intuitively, why do we get the result we do?

3. Repeat this problem for a 6 × 10 inch piece of paper.

$V = (10-2x)(6-2x)x + x^3 = (60 - 32x + 4x^2)x + x^3$

$= 60x - 32x^2 + 4x^3 + x^3$

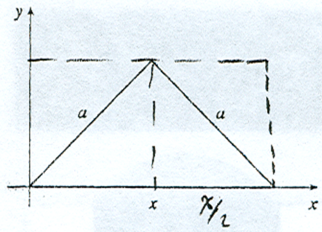
$= 60x - 32x^2 + 5x^3$

$V' = 60 - 64x + 15x^2 = 0$

$x_{max} = 1.39$, $y_{max} = \boxed{V_{max} = 35 \text{ inch}^3}$

Group Work 3, Section 4.6
Optimizing How to Solve It

1. You are given an isosceles triangle with two equal sides of length a. What is the value of the length x of the third side in order that the area of the triangle is maximal? Describe the triangle and determine the maximal area.



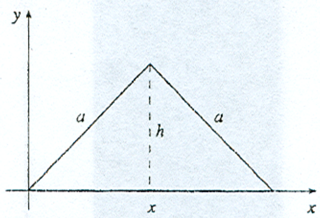
$$a^2 = h^2 + \left(\frac{x}{2}\right)^2$$

$$h^2 = a^2 - \frac{x^2}{4}$$

$$h = \sqrt{a^2 - \frac{1}{4}x^2}$$

$$A = \frac{1}{2}x\sqrt{a^2 - \frac{1}{4}x^2} = \frac{1}{2}x\left(a^2 - \frac{1}{4}x^2\right)^{+1/2}$$

2. Suppose we look at Problem 1 slightly differently. Consider the following picture of the same triangle.



$$h = \left(a^2 - \frac{1}{4}x^2\right)^{1/2}$$

(a) Write the area as a function of the altitude h.

$$A = \frac{1}{2}x\sqrt{a^2 - \frac{1}{4}x^2}$$

(b) What are the possible values for h? What is the maximal possible value?

Max " h " occur $x = 0$ at $\boxed{x = a}$

$$h = \frac{1}{2}\left(a^2 - \frac{1}{4}x^2\right)^{-1/2} \cdot \left(-\frac{2}{4}x\right) = \frac{-1x}{4\left(a^2 - \frac{1}{4}x^2\right)^{1/2}} = 0$$

(c) Without doing any computation, what triangle gives you maximal area? Compare your answer to the answer obtained in Problem 1.

See attached Page

3. Given a triangle with two sides of length 4 and 5, what should be the length of the third side in order that the area of the triangle is maximal?

See the reverse side of the attached page

$$\frac{dA}{dx} = \frac{1}{2}\left(a^2-\frac{1}{4}x^2\right)^{1/2} + \frac{x}{2}\left(\frac{1}{2}\right)\left(a^2-\frac{1}{4}x^2\right)^{-1/2}\left(-\frac{2x}{4}\right)$$

$$= \frac{\left(a^2-\frac{1}{4}x^2\right)^{1/2}}{2} - \frac{x^2}{8}\left(a^2-\frac{1}{4}x^2\right)^{-1/2} = 0$$

$$= \frac{4\left(a^2-\frac{1}{4}x^2\right)-x^2}{8\left(a^2-\frac{1}{4}x^2\right)^{1/2}}$$

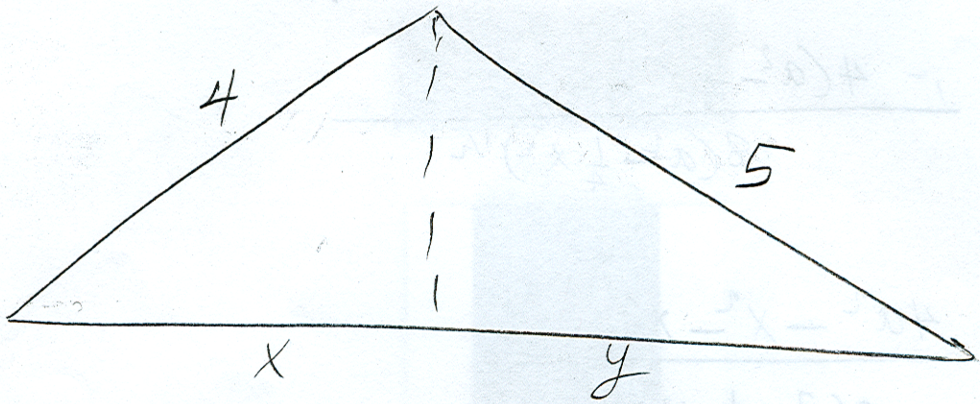
$$= \frac{4a^2-x^2-x^2}{8\left(a^2+\frac{1}{4}x^2\right)^{1/2}} = 0$$

$$-2x^2 = -4a^2$$

$$2x^2 = 4a^2 \Rightarrow x^2 = 2a^2$$

$$x = \pm\sqrt{2a^2} = \pm\sqrt{2}\,a$$

$$x = \sqrt{2}\,a$$



$$x^2 + h^2 = 4^2$$

$$y^2 + h^2 = 5^2$$

$$h^2 = 4^2 - x^2$$

$$h^2 = 5^2 - y^2$$

$$16 - x^2 = 25 - y^2$$

$$y^2 - x^2 = 25 - 16$$

$$y^2 - x^2 = 9$$

$$y^2 = 9 + x^2$$

$$\boxed{y = \sqrt{9+x^2}} \text{ and } \boxed{h = \sqrt{16-x^2}}$$

$$A = \frac{1}{2}(x+y)(h)$$

$$\boxed{A = \frac{1}{2}\left(x + \sqrt{9+x^2}\right)\left(\sqrt{16-x^2}\right)}$$

Now find $A'(x)$ and let it equal to zero. and solve for x