MA 182 Test #3 – Spring 2009

Dr. Katiraie (100 points + 10 Points Extra Credit) **Name** Solution

Show all of your work on the test paper. Full credit is not given unless the answer follows from the work shown.

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ $\sin 2\theta = 2\sin\theta\cos\theta$

1. Evaluate the following indefinite integrals. (4 Points Each)

(a) $\int \frac{dx}{x^2-4} =$

$$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$A\Big|_{x=-2} = \frac{1}{-2-2} = -1/4$$

$$B\Big|_{x=2} = \frac{1}{2+2} = \frac{+1}{4}$$

$$= \int \frac{-1/4}{x+2}\,dx + \int \frac{+1/4}{x-2}\,dx$$

$$= -\frac{1}{4}\ln(x+2) + \frac{1}{4}\ln(x-2) + c$$

$$= \ln\left(\frac{(x-2)^{1/4}}{(x+2)^{1/4}}\right) + c = \boxed{\frac{1}{4}\ln\left(\frac{x-2}{x+2}\right) + c}$$

(b) $\int \cos^2(6x)\,dx = \frac{1}{2}\int (1+\cos(12x))\,dx$

$$= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{12}\sin(12x) + c = \boxed{\frac{1}{2}x + \frac{1}{24}\sin(12x) + c}$$

(c) $\int \frac{5e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$$u = \sqrt{x}$$

$$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

$$(2)(5)\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = 10\int e^u\,du = \boxed{10e^{\sqrt{x}} + c}$$

2. Evaluate the following improper integral or show that it diverges.
Show all of your work on the test paper. Full credit is not given unless the answer follows from the work shown.

$$\int_{-2}^{5} \frac{1}{x^3}\,dx$$ (5 points)

$$\lim_{t\to 0^-}\int_{-2}^{t} \frac{1}{x^3}\,dx + \lim_{t\to 0^+}\int_{t}^{5} \frac{1}{x^3}\,dx$$

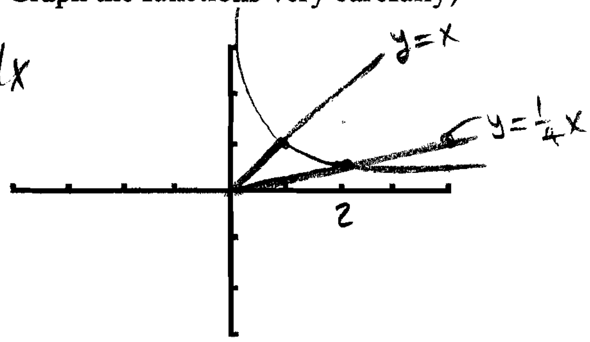
$$= \left.\frac{x^{-2}}{-2}\right|_{-2}^{t\to 0^-} + \left.\frac{x^{-2}}{-2}\right|_{t\to 0^+}^{5} = \left.\frac{-1}{2}\left(\frac{1}{x^2}\right)\right|_{-2}^{t\to 0^-} + \left.\frac{-1}{2}\left(\frac{1}{x^2}\right)\right|_{t\to 0^+}^{5}$$

$$= \frac{-1}{2}\left(+\infty - \frac{1}{4}\right) + \frac{-1}{2}\left(\frac{1}{25} - \infty\right) = \infty$$ $\boxed{\text{Diverges}}$

3. A) Set up, but do not integrate or evaluate, integral(s) which represents the area of the region R between the graphs of $y=\frac{1}{x}$, $y=x$, $y=\frac{1}{4}x$, and $x>0$ (4 points)

(Hint: Graph the functions very carefully)

$$A = \int_{x=0}^{x=1} \left(x - \frac{1}{4}x\right)dx + \int_{x=1}^{x=2} \left(\frac{1}{x} - \frac{1}{4}x\right)dx$$



3. B) Set up, but do not integrate or evaluate, the integral which represents the volume of the solid of revolution obtained by rotating region R about the x axis. (4 points)

$$\text{Volume} = \int_{x=0}^{x=1} \left(\pi x^2 - \pi\left(\frac{1}{4}x\right)^2\right)dx + \int_{x=1}^{x=2} \left(\pi\left(\frac{1}{x}\right)^2 - \pi\left(\frac{1}{4}x\right)^2\right)dx$$

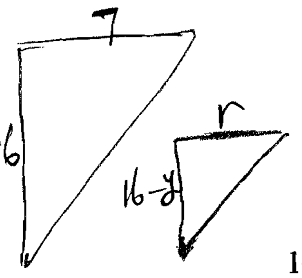
4. A tank in the shape of an inverted cone contains some water. The tank has diameter 14 feet at the top and is 16 feet deep (See figure). The water is 8 feet deep and has density $\rho = 62.5\frac{lb}{ft^3}$

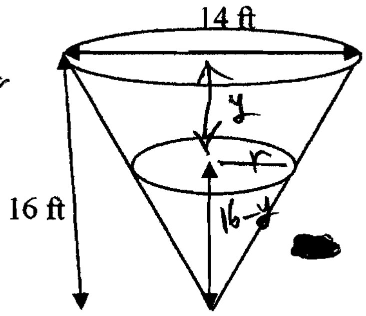
Write an integral that represents the work required to pump all the water over the top rim. Be sure to draw a picture showing how you are setting the problem up. Do not evaluate the integral.

(10 points)

$$\frac{7}{16} = \frac{r}{16-y}$$

$$r = \frac{7}{16}(16-y)$$





$$\frac{7}{16} = \frac{r}{16-y}$$

$$r = \frac{7}{16}(16-y)$$

$$A = \pi r^2 = \pi\left(\tfrac{7}{16}(16-y)\right)^2$$

$$V = \pi\left(\frac{49}{256}(16-y)^2\right)\Delta y$$

$$\text{Weight} = 62.5\pi\left(\frac{49}{256}\right)(16-y)^2\,\Delta y$$

$$\text{Work} = \int_8^{16} 62.5\pi\left(\frac{49}{256}\right)(16-y)^2\, y\, dy \quad (\text{ft-lbs})$$

5. The velocity V at time t of a point moving along a coordinate line is $V = te^{-5t}$ ft/sec. If the point is at the origin at time t = 0, find a formula for its position s at time t.

(10 points)

$$s(t) = \int t e^{-5t}\, dt \qquad s(0) = 0$$

$$u = t \qquad dv = e^{-5t}\, dt$$

$$du = dt \qquad v = \frac{-1}{5}e^{-5t}$$

$$s(t) = \frac{-1}{5}t e^{-5t} - \int \frac{-1}{5}e^{-5t}\, dt$$

$$s(t) = \frac{-1}{5}t e^{-5t} + \frac{1}{5}\cdot\frac{-1}{5}e^{-5t} + c \qquad \text{but } s(0) = 0$$

$$0 = \frac{-1}{5}(0)e^{0} - \frac{1}{25}e^{0} + c \Rightarrow c = \frac{1}{25}$$

$$s(t) = \frac{-1}{5}t e^{-5t} - \frac{1}{25}e^{-5t} + \frac{1}{25} \text{ feet}$$

6. An object is thrown vertically upward from a height of 10 ft with a velocity of 47 ft/sec. Use $a = -32\ ft/\sec^2$. $v(t) = -32t + 47$ $s(t) = -16t^2 + 47t + 10$ (8 points)

(*) a. When will the object be 30 feet above the ground?

$-16t^2 + 47t + 10 = 30 \Rightarrow -16t^2 + 47t - 20 = 0$

$t = \frac{-47 \pm \sqrt{47^2 - 4(-16)(-20)}}{-32} = \begin{cases} \to 0.52 \text{ seconds} \\ \to 2.42 \text{ seconds} \end{cases}$

b. Find the maximum height of the object analytically.

$-32t + 47 = 0 \Rightarrow t = 1.47$ sec $\boxed{y_{max} = 44.52 \text{ feet}}$

c. Find when the object hits the ground.

$-16t^2 + 47t + 10 = 0$

$t = \frac{-47 \pm \sqrt{47^2 - 4(-16)(10)}}{-32} \approx \boxed{3.14 \text{ seconds}}$

d. What is the velocity of the object just before it hits the ground?

$v(3.14) = -32(3.14) + 47 = \boxed{-53.48 \text{ ft/sec}}$

7. Given $f(x) = \ln x$

a. Find the average value of f(x) on the interval [1, 4] (7 points)

$f_{avg} = \frac{1}{3}\int_1^4 \ln x\, dx$ $u = \ln x$ $dv = dx$ $du = \frac{1}{x}dx$ $v = x$

$= \frac{1}{3}\left[x\ln x - \int \frac{1}{x} x\, dx\right] = \frac{1}{3}\left[x\ln x - x\right]\Big|_1^4$

$= \frac{1}{3}\left[4\ln 4 - 4 - (\cancel{1\ln 1}^{0} - 1)\right] = \frac{1}{3}\left[4\ln 4 - 4 + 1\right] = \boxed{\frac{1}{3}(4\ln 4 - 3)}$

b. Find c such that $f_{avg} = f(c)$ (3 points)

$f(c) = \ln c = \frac{1}{3}(4\ln 4 - 3)$

$\boxed{c = e^{\frac{1}{3}(4\ln 4 - 3)}} = e^{\frac{4}{3}\ln 4 - 1}$

$= e^{-1} e^{\ln 4^{4/3}} = \frac{4^{4/3}}{e} = \boxed{\frac{2^{\frac{8}{3}}}{e}}$

8. A spring has natural length of 2 ft. If a 20 lb. force is required to keep it stretched to a length of 2.5 ft, how much work is done in stretching it from 5 ft to 7 ft?

(6 points)

$F = K \Delta x$

$20 = K(0.5) \Rightarrow K = \frac{20}{0.5} = 40 \ \frac{lbs}{ft}$

Force $= 40x$

Work $= \int_3^5 40x\,dx = 40\frac{x^2}{2}\Big|_3^5 = 20x^2\Big|_3^5$

$7-2=5$

$5-2=3$

$= 20(25-9) = \boxed{320 \text{ ft-lbs}}$

9. Solve the following differential equations:

a) $\frac{dy}{d\theta} = \frac{e^y \sin^2\theta}{y\sec\theta}$

(7 points)

$u = \sin\theta \quad du = \cos\theta\, d\theta$

$\frac{y\,dy}{e^y} = \frac{\sin^2\theta}{\sec\theta}\,d\theta \Longrightarrow e^{-y}y\,dy = \sin^2\theta\cos\theta\,d\theta$

$u = y \quad dv = e^{-y}dy$

$du = dy \quad v = -e^{-y}$

$-e^{-y}y - \int -e^{-y}dy = \frac{\sin^3\theta}{3} + c$

$\boxed{-e^{-y}y - e^{-y} = \frac{\sin^3\theta}{3} + c}$

OR $-e^{-y}(y+1) = \frac{\sin^3\theta}{3} + c$

$\boxed{-\frac{(y+1)}{e^y} = \frac{\sin^3\theta}{3} + c}$

b) $(3y^5 + 2y)y' = x\sin x$

(6 points)

$(3y^5 + 2y)dy = x\sin x\,dx$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$\frac{3y^6}{6} + \frac{2y^2}{2} = -x\cos x - \int -\cos x\,dx$

$\boxed{\frac{y^6}{2} + y^2 = -x\cos x + \sin x + c}$

10. A roast is taken from the refrigerator (where the temperature is 40° F) and put in a 370° F oven. One hour later, the meat thermometer shows a temperature of 100° F. Assuming that the roast heats at a rate which obeys Newton's Law of Heating and Cooling, that is, that the rate of heating or cooling of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings, the differential equation that models this situation is

$$\frac{dT}{dt} = k(370 - T)$$

where T is the temperature of the object at time t.

(a) Solve this differential equation subject to the conditions given. Find the value of all constants. (8 points)

$T(0) = 40°F$

$$\frac{dT}{370 - T} = k\,dt$$

$$-\ln(370 - T) = kt + c \Rightarrow -\ln(370 - 40) = k(0) + c \Rightarrow \boxed{c = -\ln(330)}$$

$$-\ln(370 - T) = kt - \ln 330 \qquad \text{But } T(1) = 100$$

$$-\ln(370 - 100) = k(1) - \ln 330 \Rightarrow -\ln(270) = k - \ln 330 \Rightarrow \boxed{k = \ln\left(\frac{330}{270}\right)}$$

$$-\ln(370 - T) = \ln\left(\frac{330}{270}\right)t - \ln 330 \Rightarrow \ln(370 - T) = -\ln\left(\frac{330}{270}\right)t + \ln 330$$

$$370 - T = 330\, e^{-\ln\left(\frac{330}{270}\right)t} \Rightarrow \boxed{T = -330\, e^{-\ln\left(\frac{330}{270}\right)t} + 370}$$

(b) If the roast is done when the internal temperature reaches 140° F, what is the total time the roast should be in the oven? (2 points)

$$140 = -330\, e^{-\ln\left(\frac{330}{270}\right)t} + 370$$

$$140 - 370 = -330\, e^{-\ln\left(\frac{330}{270}\right)t}$$

$$\frac{-230}{-330} = e^{-\ln\left(\frac{330}{270}\right)t}$$

$$\ln\left(\frac{230}{330}\right) = \cancel{\ln e}^{-\ln\left(\frac{330}{270}\right)t}$$

$$\ln\left(\frac{230}{330}\right) = -\ln\left(\frac{330}{270}\right)t \Rightarrow t = \frac{\ln\left(\frac{230}{330}\right)}{-\ln\left(\frac{330}{270}\right)} \boxed{\cong 1.8 \text{ HRS}}$$

11. Cobalt-60 has a half-life of 5.24 years. (Hint: $\frac{dy}{dt} = ky$, and $y(t) = y_0 e^{kt}$)

a) Find the value of k. (2 points)

$$\frac{1}{2} = 1e^{5.24K}$$

$$\ln \tfrac{1}{2} = \cancel{\ln} e^{5.24K} \implies \boxed{K = \frac{\ln(1/2)}{5.24}}$$

b) Find the mass that remains from a 100 mg sample after 20 years. (3 points)

$$y = 100\, e^{\frac{\ln(1/2)}{5.24} * 20} = \boxed{7.096 \text{ mg}}$$

c) How long would it take for the mass to decay to 1 mg? (3 points)

$$1 = 100\, e^{\frac{\ln(1/2)}{5.24} t}$$

$$\frac{1}{100} = e^{\frac{\ln(1/2)}{5.24} t}$$

$$\ln\left(\frac{1}{100}\right) = \ln e^{\frac{\ln(1/2)}{5.24} t}$$

$$\ln\left(\frac{1}{100}\right) = \frac{\ln(1/2)}{5.24} t$$

$$t = \frac{\ln(1/100)}{\left(\frac{\ln(1/2)}{5.24}\right)} = \boxed{34.8 \text{ years}}$$

Test III Avg = 87.5
14A
5B
1C
2D
3F
[redacted]

Extra Credit

Solve the following initial-value problems
(Please isolate y)

a) $\frac{dy}{dt} + 2yt = y \qquad y(0)=6$ (4 points)

$\frac{dy}{dt} = y - 2yt \overset{\text{Factor}}{\Longrightarrow} \frac{dy}{dt} = y(1-2t)$

$\frac{dy}{y} = (1-2t)dt$

$\ln y = t - \frac{2t^2}{2} + c$ but $y(0)=6 \Rightarrow \boxed{\ln 6 = c}$

$\ln y = t - t^2 + \ln 6$

$y = e^{t-t^2+\ln 6} = \boxed{6e^{t-t^2}}$

b) $(1+\cos x)y' = ((1+e^{-y})\sin x \qquad y(0)=0$ (6 points)

$(1+\cos x)\frac{dy}{dx} = (1+e^{-y})\sin x$

$\frac{dy}{1+e^{-y}} = \frac{\sin x}{1+\cos x}dx$

$\frac{dy}{\frac{e^y+1}{e^y}} = \frac{\sin x}{1+\cos x}dx$

$\frac{e^y\,dy}{e^y+1} = \frac{\sin x}{1+\cos x}dx$

$\ln|e^y+1| = -\ln|1+\cos x| + c$

$\ln(e^0+1) = -\ln(1+\cos 0) + c \Rightarrow c = \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

$\ln(e^y+1) = -\ln(1+\cos x) + \ln 4$

$\ln(e^y+1) = \ln\left(\frac{4}{1+\cos x}\right)$

$e^y + 1 = \frac{4}{1+\cos x}$

$e^y = \frac{4}{1+\cos x} - 1$

$\boxed{y = \ln\left(\frac{4}{1+\cos x} - 1\right)}$ OR

$y = \ln\left(\frac{4-1-\cos x}{1+\cos x}\right) = \boxed{\ln\left(\frac{3-\cos x}{1+\cos x}\right)}$