Applications of Differentiation

4

4.5 Indeterminate Forms and l'Hospital's Rule

Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x* = 1, we need to know how *F* behaves *near* 1. In particular, we would like to know the value of the limit

1 $$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply law of limits (the limit of a quotient is the quotient of the limits) because the limit of the denominator is 0.

Indeterminate Forms and l'Hospital's Rule

In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\mathbf{\frac{0}{0}}$.

Indeterminate Forms and l'Hospital's Rule

For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as *l'Hospital's Rule,* for the evaluation of indeterminate forms.

Indeterminate Forms and l'Hospital's Rule

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of *F* and need to evaluate the limit

$$\lim_{x \to \infty} \frac{\ln x}{x - 1} \qquad \boxed{2}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

Indeterminate Forms and l'Hospital's Rule

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type** ∞/∞.

This type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of *x* that occurs in the denominator.

Indeterminate Forms and l'Hospital's Rule

For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

Indeterminate Forms and l'Hospital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example 1 – *An Indeterminate Form of Type 0/0*

Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Solution:

Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Indeterminate Products

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the limit will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number.

This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Example 6

Evaluate $\lim_{x \to 0^+} x \ln x$. Use the knowledge of this limit, together with information from derivatives, to sketch the curve $y = x \ln x$.

Solution:

The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$.

Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

Example 6 – *Solution*

cont'd

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} (-x) = 0$$

If $f(x) = x \ln x$, then

$$f'(x) = x \cdot \frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

so $f'(x) = 0$ when $\ln x = -1$, which means that $x = e^{-1}$.

Example 6 – *Solution*

cont'd

In fact, $f'(x) > 0$ when $x > e^{-1}$ and $f'(x) < 0$ when $x < e^{-1}$, so f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$. Thus, by the First Derivative Test, $f(1/e) = -1/e$ is a local (and absolute) minimum.

Also, $f''(x) = 1/x > 0$, so f is concave upward on $(0, \infty)$.

We use this information, together with the crucial knowledge that $\lim_{x\to 0^+} f(x) = 0$, to sketch the curve in Figure 5.

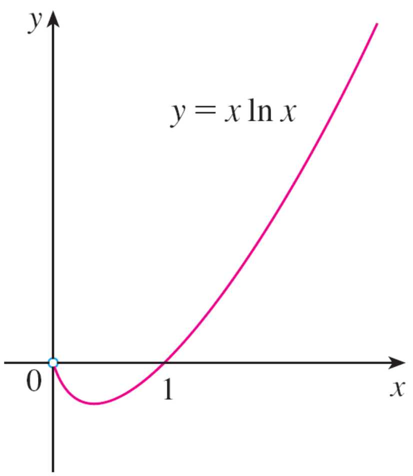

Figure 5

Indeterminate Differences

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**.

Again there is a contest between *f* and *g*. Will the answer be ∞ (*f* wins) or will it be $-\infty$ (*g* wins) or will they compromise on a finite number?

To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .

Example 7 – *An Indeterminate Form of Type* $\infty - \infty$

Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

Solution:

First notice that sec $x \to \infty$ and tan $x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x}$$

$$= \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

Indeterminate Powers

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0

2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0

3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Indeterminate Powers

Each of these three cases can be treated either by taking the natural logarithm:

let $y = [f(x)]^{g(x)}$, then $\ln y = g(x) \ln f(x)$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

Example 8 – *An Indeterminate Form of Type* 1^{∞}

Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

Solution:

First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate.

Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then

$$\ln y = \ln [(1 + \sin 4x)^{\cot x}]$$
$$= \cot x \ln (1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

Example 8 – *Solution*

cont'd

$$= \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x}$$

$= 4$

So far we have computed the limit of ln *y*, but what we want is the limit of *y*.

To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y$$

$$= \lim_{x \to 0^+} e^{\ln y}$$

$$= e^4$$