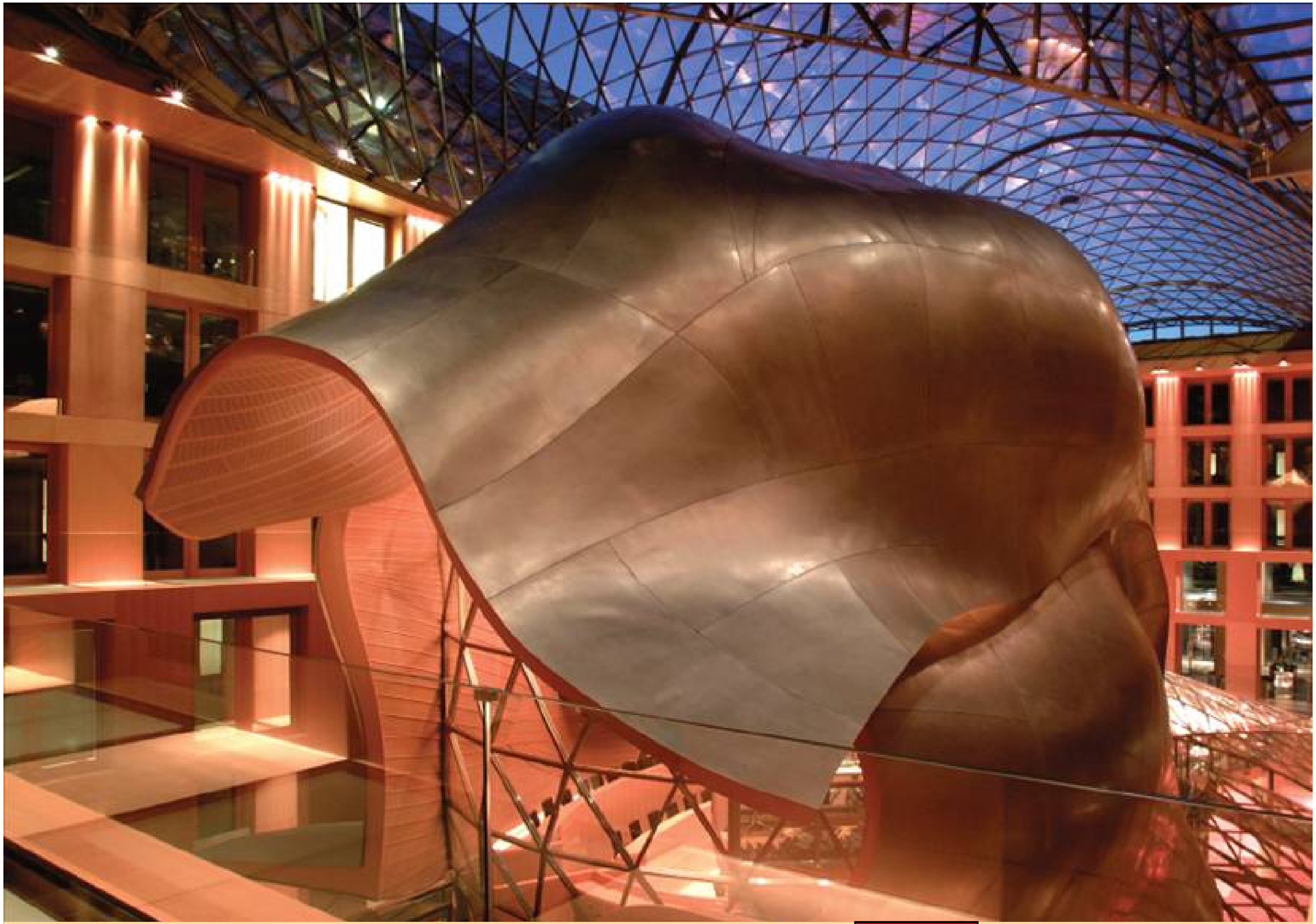

Integrals

5

5.3 Evaluating Definite Integrals

Evaluating Definite Integrals

We have computed integrals from the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult.

Sir Isaac Newton discovered a much simpler method for evaluating integrals and a few years later Leibniz made the same discovery.

They realized that they could calculate $\int_a^b f(x)\,dx$ if they happened to know an antiderivative *F* of *f*.

Evaluating Definite Integrals

Their discovery, called the Evaluation Theorem, is part of the Fundamental Theorem of Calculus.

Evaluation Theorem If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

This theorem states that if we know an antiderivative *F* of *f*, then we can evaluate $\int_a^b f(x)\,dx$ simply by subtracting the values of *F* at the endpoints of the interval [*a*, *b*].

Evaluating Definite Integrals

It is very surprising that $\int_a^b f(x)\,dx$, which was defined by a complicated procedure involving all of the values of $f(x)$ for $a \le x \le b$, can be found by knowing the values of $F(x)$ at only two points, a and b.

For instance, we know that an antiderivative of the function $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, so the Evaluation Theorem tells us that

$$\int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3} \cdot 1^3 - \tfrac{1}{3} \cdot 0^3 = \tfrac{1}{3}$$

Although the Evaluation Theorem may be surprising at first glance, it becomes plausible if we interpret it in physical terms.

Evaluating Definite Integrals

If $v(t)$ is the velocity of an object and $s(t)$ is its position at time t, then $v(t) = s'(t)$, so s is an antiderivative of v.

We have considered an object that always moves in the positive direction and made the guess that the area under the velocity curve is equal to the distance traveled. In symbols:

$$\int_a^b v(t)\, dt = s(b) - s(a)$$

That is exactly what the Evaluation Theorem says in this context.

Evaluating Definite Integrals

When applying the Evaluation Theorem we use the notation

$$F(x)\Big]_a^b = F(b) - F(a)$$

and so we can write

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \qquad \text{where} \qquad F' = f$$

Other common notations are $F(x)\big|_a^b$ and $[F(x)]_a^b$.

Example 1 – *Using the Evaluation Theorem*

Evaluate $\int_1^3 e^x\,dx$.

Solution:

An antiderivative of $f(x) = e^x$ is $F(x) = e^x$, so we use the Evaluation Theorem as follows:

$$\int_1^3 e^x\,dx = e^x\Big]_1^3$$

$$= e^3 - e$$

Indefinite Integrals

Indefinite Integrals

We need a convenient notation for antiderivatives that makes them easy to work with.

Because of the relation given by the Evaluation Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

Indefinite Integrals

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number,* whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions).

The connection between them is given by the Evaluation Theorem: If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

Indefinite Integrals

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant. For instance, the formula

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

is valid (on any interval that doesn't contain 0) because $(d/dx) \ln|x| = 1/x$.

So an indefinite integral $\int f(x)\,dx$ can represent either a particular antiderivative of f or an entire *family* of antiderivatives (one for each value of the constant C).

Indefinite Integrals

The effectiveness of the Evaluation Theorem depends on having a supply of antiderivatives of functions.

We therefore restate the Table of Antidifferentiation Formulas, together with a few others, in the notation of indefinite integrals.

Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance,

$$\int \sec^2 x \, dx = \tan x + C \quad \text{because} \quad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

1 Table of Indefinite Integrals

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

Example 3

Find the general indefinite integral

$$\int (10x^4 - 2 \sec^2 x)\, dx$$

Solution:
Using our convention and Table 1 and properties of integrals, we have

$$\int (10x^4 - 2 \sec^2 x)\, dx = 10 \int x^4\, dx - 2 \int \sec^2 x\, dx$$

$$= 10 \frac{x^5}{5} - 2 \tan x + C$$

$$= 2x^5 - 2 \tan x + C$$

You should check this answer by differentiating it.

Applications

Applications

The Evaluation Theorem says that if *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b. [Note that y could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.]

So we can reformulate the Evaluation Theorem in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{2} \qquad \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

  is the net change of position, or *displacement,* of the particle during the time period from t_1 to t_2.

Applications

We have guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left).

In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

3
$$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Applications

Figure 4 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.

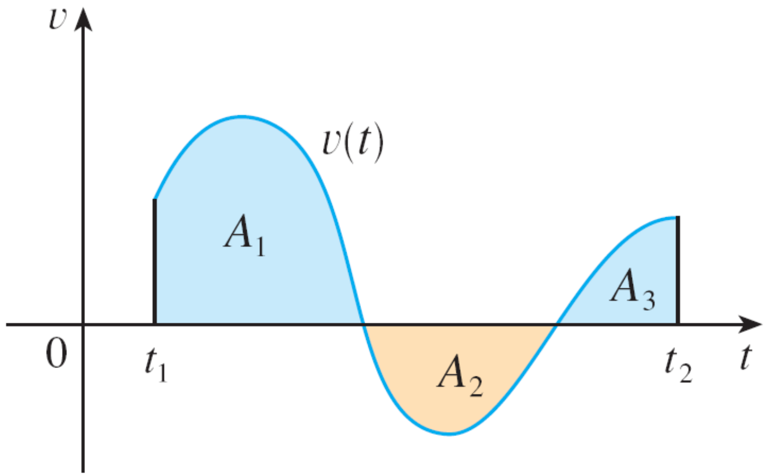

Figure 4

displacement = $\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$

distance = $\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$

Example 7 – *Displacement Versus Distance*

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Solution:

(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\, dt$$

$$= \int_1^4 (t^2 - t - 6)\, dt$$

Example 7 – *Solution*

cont'd

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$$

$$= -\frac{9}{2}$$

This means that the particle's position at time $t = 4$ is 4.5 m to the left of its position at the start of the time period.

(b) Note that $v(t) = t^2 – t – 6 = (t – 3)(t + 2)$ and so $v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4].

Example 7 – *Solution*

cont'd

Thus, from Equation 3, the distance traveled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6}$$

$$\approx 10.17 \text{ m}$$