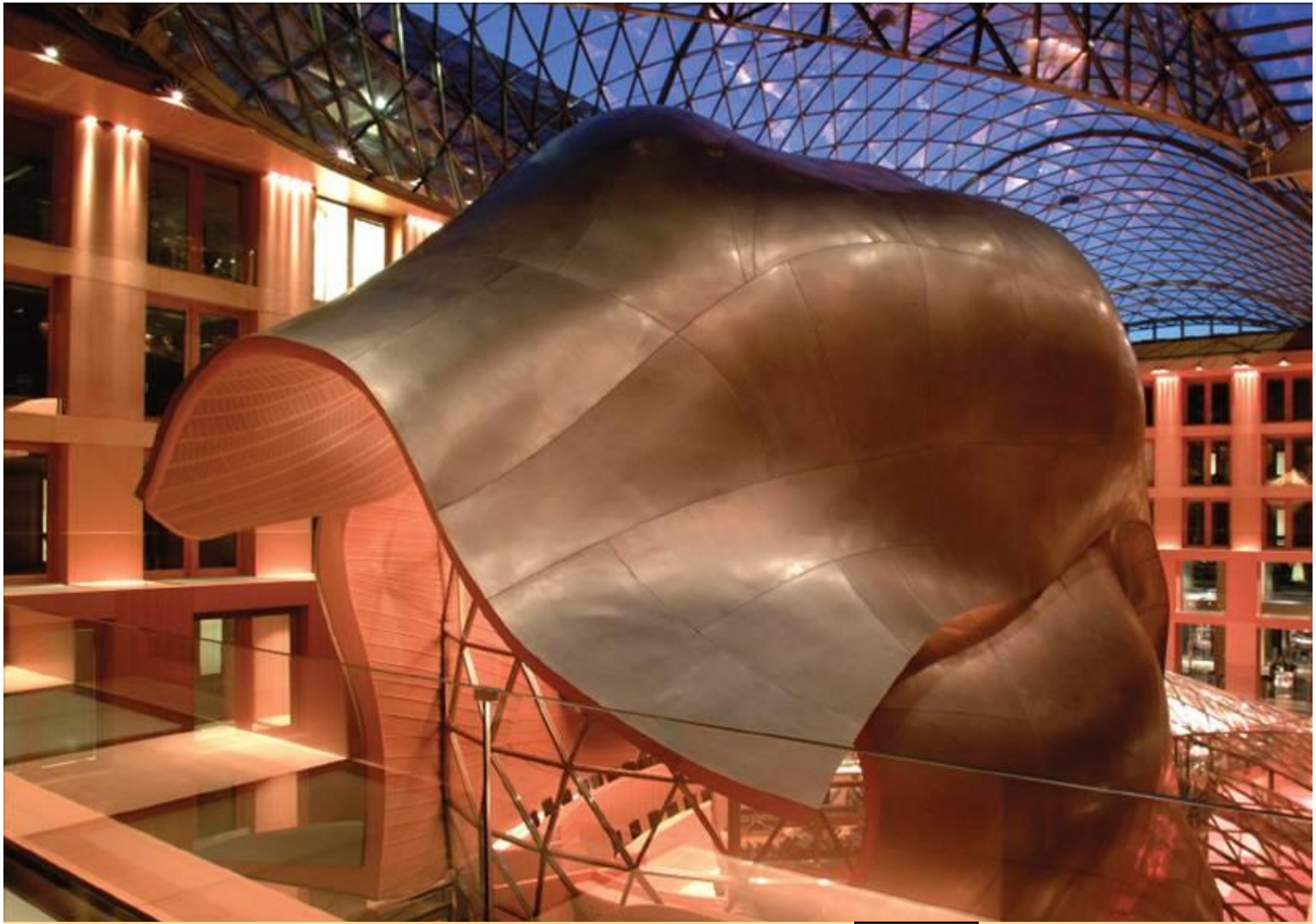

Integrals

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5.6 Integration by Parts

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integration by Parts

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{1} \qquad \boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx}$$

Formula 1 is called the **formula for integration by parts**.

Integration by Parts

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

2 $$\boxed{\int u\,dv = uv - \int v\,du}$$

Example 1 – *Integrating by Parts*

Find $\int x \sin x \, dx$.

Solution Using Formula 1:

Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}\int x \sin x \, dx &= f(x)g(x) - \int g(x)f'(x) \, dx \\ &= x(-\cos x) - \int (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C\end{aligned}$$

Example 1 – *Solution*

cont'd

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

Solution Using Formula 2:

Let

$$u = x \qquad\qquad dv = \sin x\, dx$$

Then $$du = dx \qquad\qquad v = -\cos x$$

and so

$$\int x \sin x\, dx = \int \overbrace{x}^{u}\, \overbrace{\sin x\, dx}^{dv}$$

Example 1 – *Solution*

cont'd

$$= \underbrace{x}_{u}\,\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{u}\,\underbrace{dx}_{du}$$

$$= -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C$$

Integration by Parts

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts.

Evaluating both sides of Formula 1 between *a* and *b*, assuming *f′* and *g′* are continuous, and using the Evaluation Theorem, we obtain

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$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$