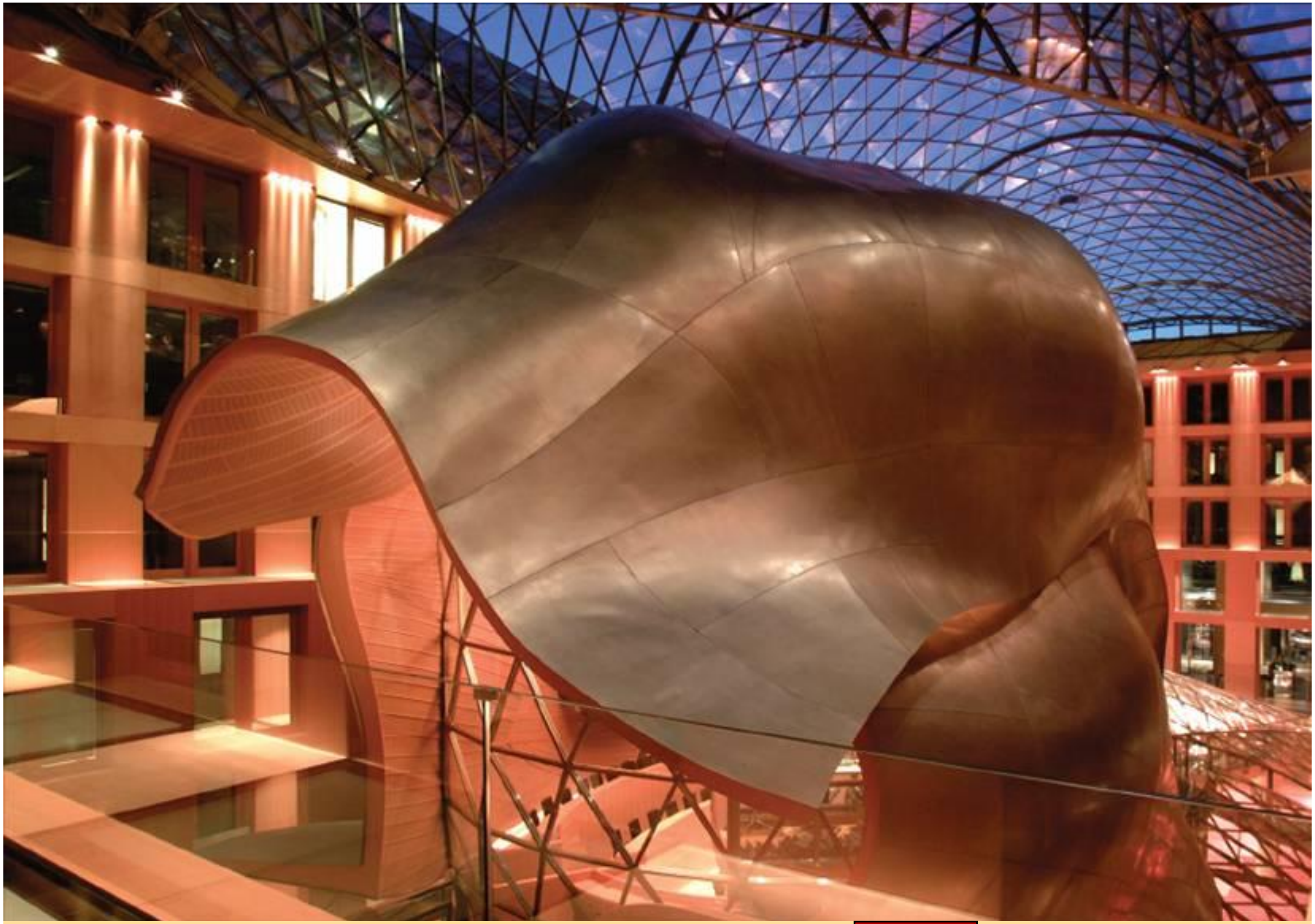

Integrals

5

5.7 Additional Techniques of Integration

Trigonometric Integrals

Trigonometric Integrals

We can use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1 – *An Integral with an Odd Power of cos x*

Evaluate $\int \cos^3 x \, dx$.

Solution:

We would like to use the Substitution Rule, but simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x \, dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. (Similarly, a power of sine would require an extra $\cos x$ factor.)

Here we separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity $\sin^2 x + \cos^2 x = 1$:

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x) \cos x$$

Example 1 – *Solution*

cont'd

We can then evaluate the integral by substituting $u = \sin x$, so $du = \cos x \, dx$ and

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (1 - u^2) du$$

$$= u - \tfrac{1}{3} u^3 + C$$

$$= \sin x - \tfrac{1}{3} \sin^3 x + C$$

Trigonometric Integrals

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine).

The identity $\sin^2 x + \cos^2 x = 1$ enables us to convert back and forth between even powers of sine and cosine.

Trigonometric Integrals

If the integrand contains only even powers of both sine and cosine, however, this strategy fails. In this case, we can take advantage of the *half-angle identities*

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

and

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

Trigonometric Substitution

Trigonometric Substitution

A number of practical problems require us to integrate algebraic functions that contain an expression of the form

$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$.

Sometimes, the best way to perform the integration is to make a trigonometric substitution that gets rid of the root sign.

Example 3

Prove that the area of a circle with radius r is πr^2.

Solution:

For simplicity, let's place the circle with its center at the origin, so its equation is $x^2 + y^2 = r^2$. Solving this equation for y, we get

$$y = \pm\sqrt{r^2 - x^2}$$

Because the circle is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2).

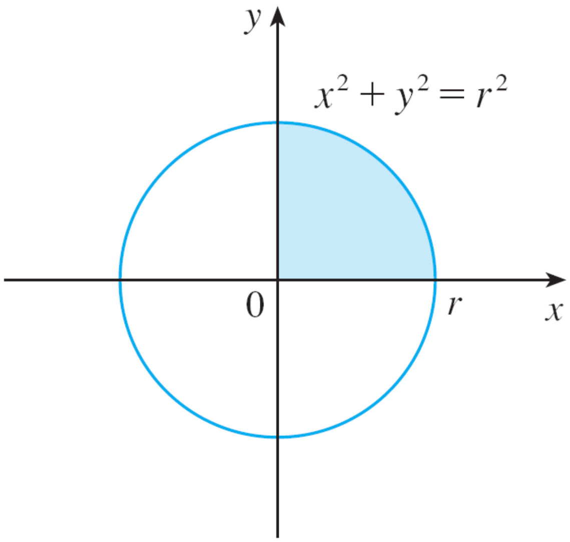

Figure 2

Example 3 – *Solution*

cont'd

The part of the circle in the first quadrant is given by the function

$$y = \sqrt{r^2 - x^2} \qquad 0 \leqslant x \leqslant r$$

and so $\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2}\, dx$

To simplify this integral, we would like to make a substitution that turns $r^2 - x^2$ into the square of something. The trigonometric identity $1 - \sin^2\theta = \cos^2\theta$ is useful here. In fact, because

$$\begin{aligned} r^2 - r^2 \sin^2\theta &= r^2(1 - \sin^2\theta) \\ &= r^2\cos^2\theta \end{aligned}$$

Example 3 – *Solution*

cont'd

We make the substitution

$$x = r \sin \theta$$

Since $0 \le x \le r$, we restrict θ so that $0 \le \theta \le \pi/2$. We have $dx = r \cos \theta \, d\theta$ and

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2 \sin^2\theta}$$

$$= \sqrt{r^2 \cos^2\theta}$$

$$= r \cos \theta$$

because $\cos \theta \ge 0$ when $0 \le \theta \le \pi/2$.

Example 3 – *Solution*

cont'd

Therefore the Substitution Rule gives

$$\int_0^r \sqrt{r^2 - x^2}\, dx = \int_0^{\pi/2} (r\cos\theta)\, r\cos\theta\, d\theta$$

$$= r^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

This trigonometric integral is similar to the one in Example 2; we integrate $\cos^2\theta$ by means of the identity

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

Example 3 – *Solution*

cont'd

Thus

$$\tfrac{1}{4}A = r^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \tfrac{1}{2} r^2 \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= \tfrac{1}{2} r^2 \Big[\theta + \tfrac{1}{2} \sin 2\theta\Big]_0^{\pi/2}$$

$$= \tfrac{1}{2} r^2 \left(\frac{\pi}{2} + 0 - 0\right)$$

$$= \tfrac{1}{4} \pi r^2$$

We have therefore proved the famous formula $A = \pi r^2$.

Partial Fractions

Partial Fractions

We integrate rational functions (ratios of polynomials) by expressing them as sums of simpler fractions, called *partial fractions*, that we already know how to integrate.

The following example illustrates the simplest case.

Example 4

Find $\int \frac{5x - 4}{2x^2 + x - 1}\, dx.$

Solution:

Notice that the denominator can be factored as a product of linear factors:

$$\frac{5x - 4}{2x^2 + x - 1} = \frac{5x - 4}{(x + 1)(2x - 1)}$$

Example 4 – *Solution*

cont'd

In a case like this, where the numerator has a smaller degree than the denominator, we can write the given rational function as a sum of partial fractions:

$$\frac{5x - 4}{(x + 1)(2x - 1)} = \frac{A}{x + 1} + \frac{B}{2x - 1}$$

where A and B are constants.

To find the values of A and B we multiply both sides of this equation by $(x + 1)(2x - 1)$, obtaining

$$5x - 4 = A(2x - 1) + B(x + 1)$$

or $$5x - 4 = (2A + B)x + (-A + B)$$

Example 4 – *Solution*

cont'd

The coefficients of x must be equal and the constant terms are also equal. So

$$2A + B = 5 \qquad \text{and} \qquad -A + B = -4$$

Solving this system of linear equations for A and B, we get $A = 3$ and $B = -1$, so

$$\frac{5x - 4}{(x + 1)(2x - 1)} = \frac{3}{x + 1} - \frac{1}{2x - 1}$$

Example 4 – *Solution*

cont'd

Each of the resulting partial fractions is easy to integrate (using the substitutions $u = x + 1$ and $u = 2x - 1$, respectively). So we have

$$\int \frac{5x - 4}{2x^2 + x - 1}\,dx = \int \left(\frac{3}{x + 1} - \frac{1}{2x - 1} \right) dx$$

$$= 3 \ln|x + 1| - \tfrac{1}{2} \ln|2x - 1| + C$$

Partial Fractions

Note 1: If the degree in the numerator in Example 4 had been the same as that of the denominator, or higher, we would have had to take the preliminary step of performing a long division. For instance,

$$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1} = x - 6 + \frac{5x - 4}{(x + 1)(2x - 1)}$$

Partial Fractions

Note 2: If the denominator has more than two linear factors, we need to include a term corresponding to each factor.

For example,

$$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

where *A*, *B*, and *C* are constants determined by solving a system of three equations in the unknowns *A*, *B*, and *C*.

Partial Fractions

Note 3: If a linear factor is repeated, we need to include extra terms in the partial fraction expansion. Here's an example:

$$\frac{x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

Partial Fractions

Note 4: When we factor a denominator as far as possible, it might happen that we obtain an irreducible quadratic factor $ax^2 + bx + c$, where the discriminant $b^2 - 4ac$ is negative.

Then the corresponding partial fraction is of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. This term can be integrated by completing the square and using the formula

$$\boxed{1} \qquad \boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$