Applications of Integration

6

6.1 More About Areas

Areas Between Curves

Areas Between Curves

Consider the region *S* that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where *f* and *g* are continuous functions and $f(x) \geq g(x)$ for all *x* in $[a, b]$. (See Figure 1.)

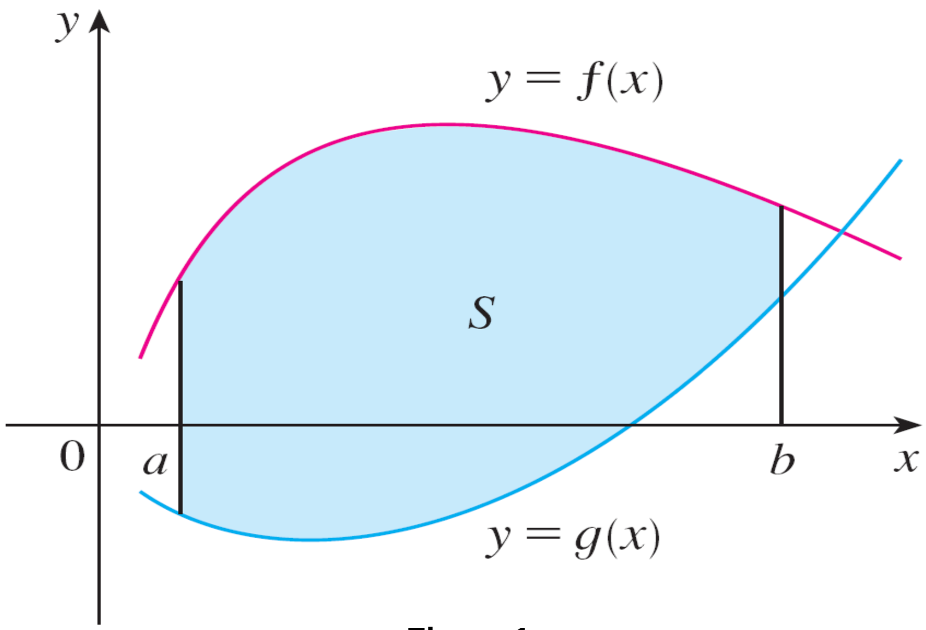
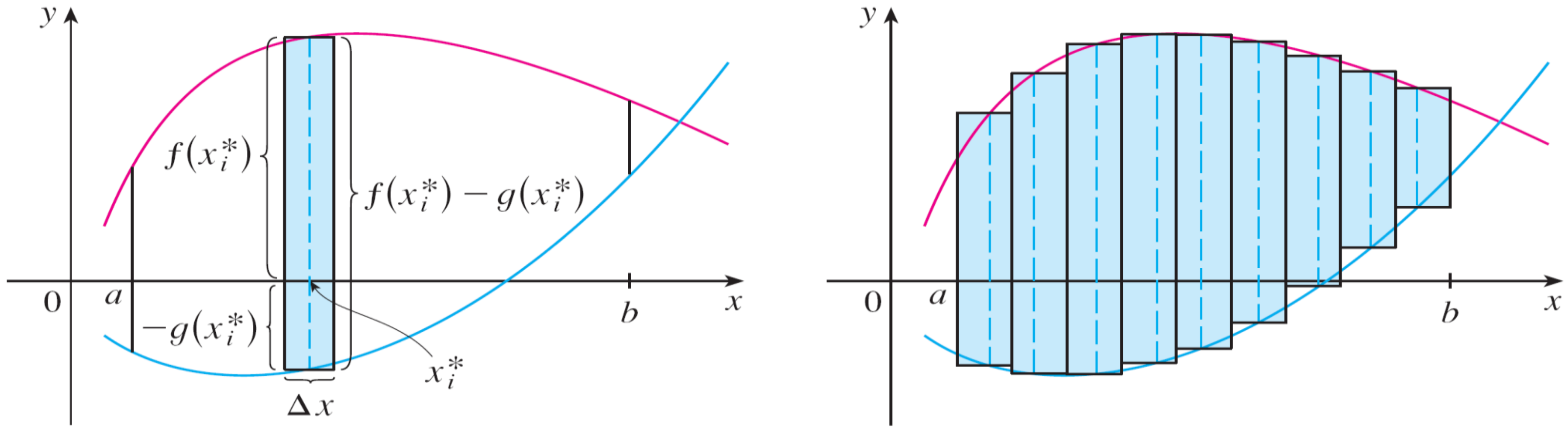


Figure 1

$S = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$

Areas Between Curves

We divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.)



(a) Typical rectangle

(b) Approximating rectangles

Figure 2

Areas Between Curves

The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

is therefore an approximation to what we intuitively think of as the area of *S*.

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** *A* of the region *S* as the limiting value of the sum of the areas of these approximating rectangles.

Areas Between Curves

1

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

We recognize the limit in (1) as the definite integral of *f* – *g*. Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geqslant g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Notice that in the special case where $g(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces.

Areas Between Curves

In the case where both *f* and *g* are positive, you can see from Figure 3 why (2) is true:

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

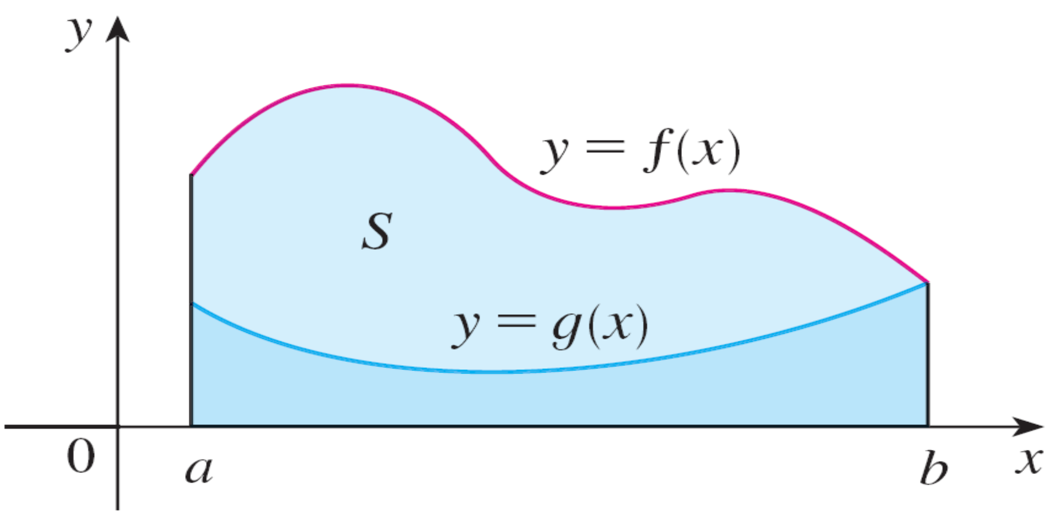


Figure 3

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Example 1 – *Area Between Two Curves*

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution:

The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$.

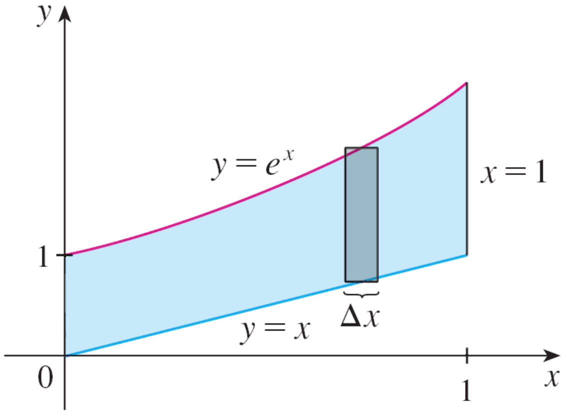


Figure 4

Example 1 – *Solution*

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So we use the area formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (e^x - x)\, dx$$

$$= e^x - \tfrac{1}{2}x^2 \Big]_0^1$$

$$= e - \tfrac{1}{2} - 1$$

$$= e - 1.5$$

Areas Between Curves

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in (1).

In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle as in Figure 5.

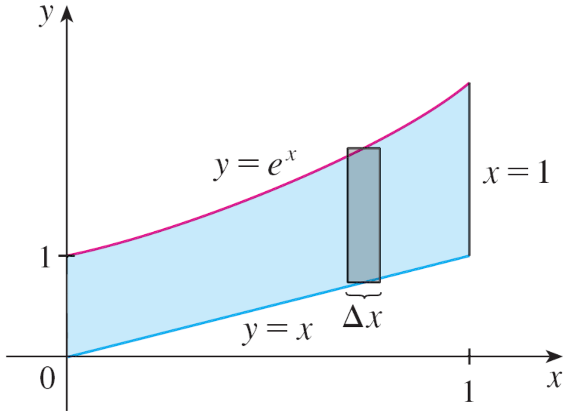

Figure 4

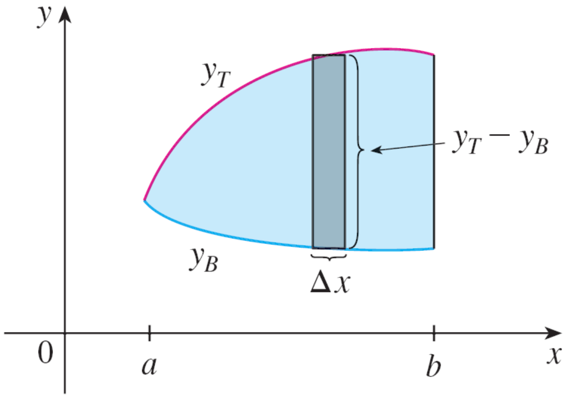

Figure 5

Areas Between Curves

Then the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$

summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point.

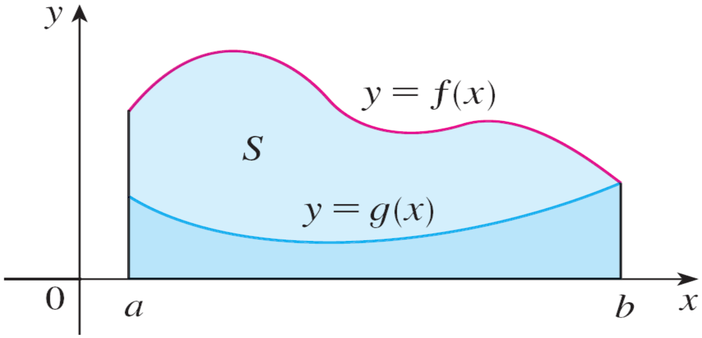

Figure 3

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Areas Between Curves

Some regions are best treated by regarding *x* as a function of *y*. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where *f* and *g* are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$

(see Figure 9), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

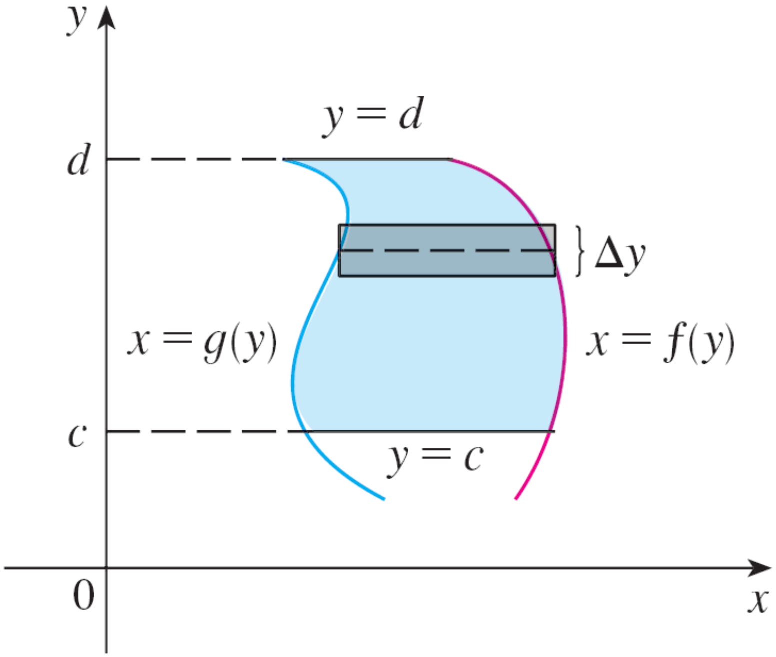

Figure 9

Areas Between Curves

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 10 illustrates, we have

$$A = \int_c^d (x_R - x_L)\, dy$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and Δy.

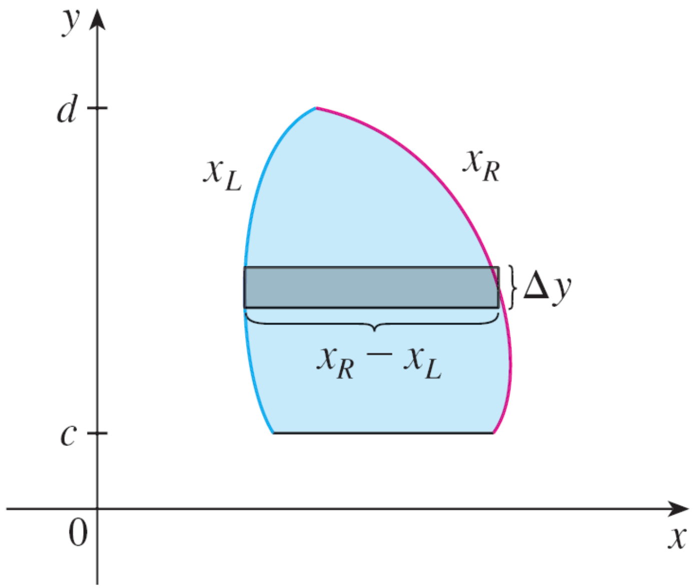

Figure 10

Areas Enclosed by Parametric Curves

Areas Enclosed by Parametric Curves

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$.

If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

Example 6

Find the area under one arch of the cycloid

$x = r(\theta - \sin\ \theta)$ $\quad\quad$ $y = r(1 - \cos\ \theta)$

(See Figure 13.)

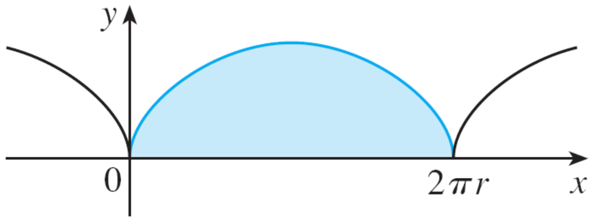

Figure 13

Solution:

One arch of the cycloid is given by $0 \le \theta \le 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\ \theta)$ and $dx = r(1 - \cos\ \theta)\ d\theta$, we have

$$A = \int_0^{2\pi r} y\, dx = \int_0^{2\pi} r(1 - \cos\ \theta)\, r(1 - \cos\ \theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} (1 - \cos\ \theta)^2\, d\theta$$

Example 6 – *Solution*

cont'd

$$= r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= r^2\left(\tfrac{3}{2} \cdot 2\pi\right)$$

$$= 3\pi r^2$$