Applications of Integration

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6.2 Volumes

Volumes

In trying to find the volume of a solid we face the same type of problem as in finding areas.

We have an intuitive idea of what volume means, but we must make this idea precise by using calculus to give an exact definition of volume.

Volumes

We start with a simple type of solid called a **cylinder** (or, more precisely, a *right cylinder*).

As illustrated in Figure 1(a), a cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane.
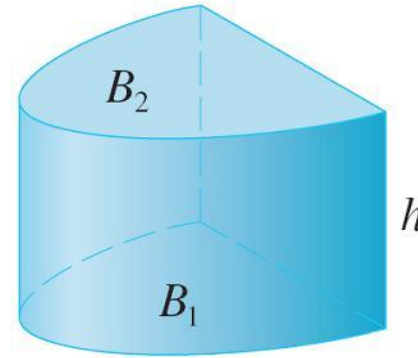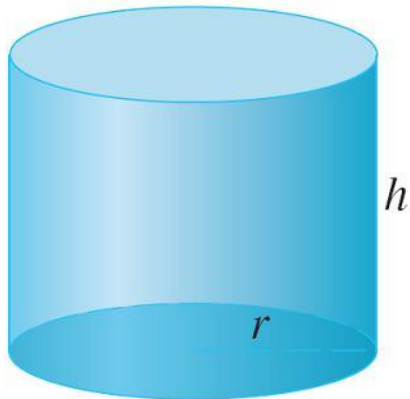

(a) Cylinder $V = Ah$

Figure 1(a)

The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume V of the cylinder is defined as
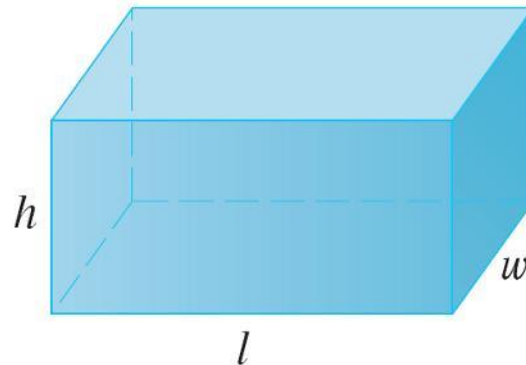
$$V = Ah$$

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Volumes

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$ [see Figure 1(b)], and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a *rectangular parallelepiped*) with volume $V = lwh$ [see Figure 1(c)].

(b) Circular cylinder $V = \pi r^2 h$

Figure 1(b)

(c) Rectangular box $V = lwh$

Figure 1(c)

Volumes

For a solid *S* that isn't a cylinder we first "cut" *S* into pieces and approximate each piece by a cylinder. We estimate the volume of *S* by adding the volumes of the cylinders. We arrive at the exact volume of *S* through a limiting process in which the number of pieces becomes large.

We start by intersecting *S* with a plane and obtaining a plane region that is called a **cross-section** of *S*.

Volumes

Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leq x \leq b$. (See Figure 2. Think of slicing S with a knife through x and computing the area of this slice.)

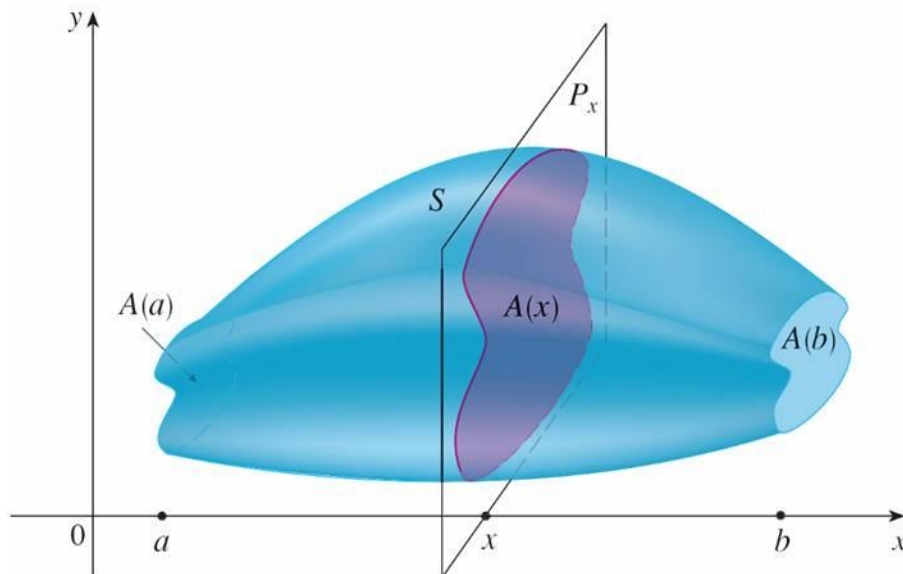The cross-sectional area $A(x)$ will vary as x increases from a to b.

Figure 2

Volumes

Let's divide S into n "slabs" of equal width Δx by using the planes P_{x_1}, P_{x_2}, . . . to slice the solid. (Think of slicing a loaf of bread.)

If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and "height" Δx. (See Figure 3.)
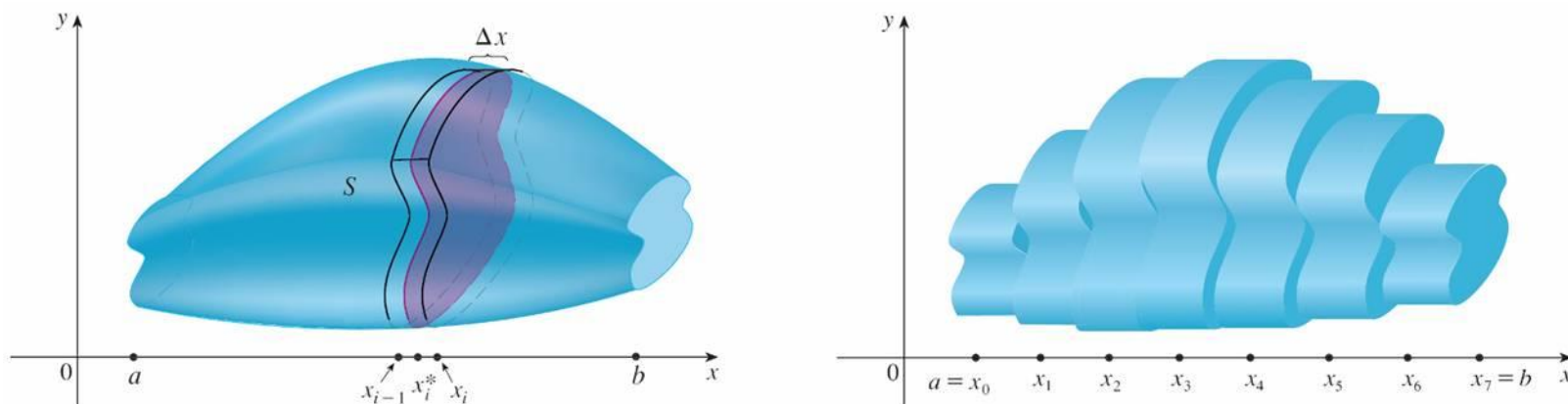

Figure 3

Volumes

The volume of this cylinder is $A(x_i^*)\, \Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\, \Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume (that is, what we think of intuitively as the volume):

$$V \approx \sum_{i=1}^{n} A(x_i^*)\, \Delta x$$

Volumes

This approximation appears to become better and better as $n \to \infty$. (Think of the slices as becoming thinner and thinner.)

Therefore we *define* the volume as the limit of these sums as $n \to \infty$.

But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

Definition of Volume   Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \, \Delta x = \int_a^b A(x) \, dx$$

Volumes

When we use the volume formula $V = \int_a^b A(x)\,dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A\,dx = A(b - a)$; this agrees with the formula $V = Ah$.

Example 1

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution:

If we place the sphere so that its center is at the origin (see Figure 4), then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is $y = \sqrt{r^2 - x^2}$.

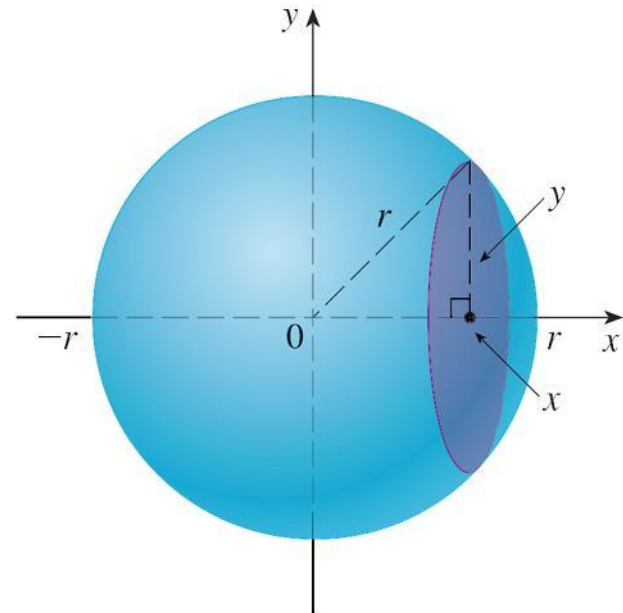So the cross-sectional area is
$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Figure 4

Example 1 – *Solution*

cont'd

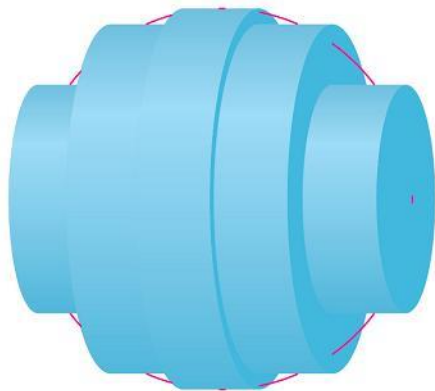Using the definition of volume with $a = -r$ and $b = r$, we have

$$V = \int_{-r}^{r} A(x)\, dx$$

$$= \int_{-r}^{r} \pi(r^2 - x^2)\, dx$$

$$= 2\pi \int_{0}^{r} (r^2 - x^2)\, dx \qquad \text{(The integrand is even.)}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_{0}^{r}$$

$$= 2\pi \left(r^3 - \frac{r^3}{3} \right)$$
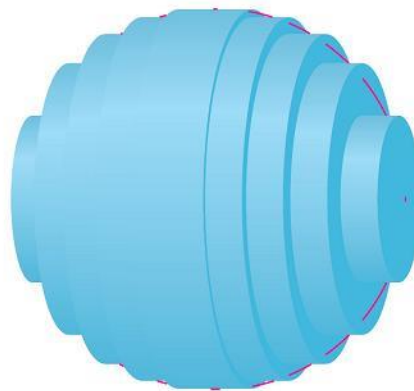
$$= \tfrac{4}{3}\pi r^3$$

Volumes

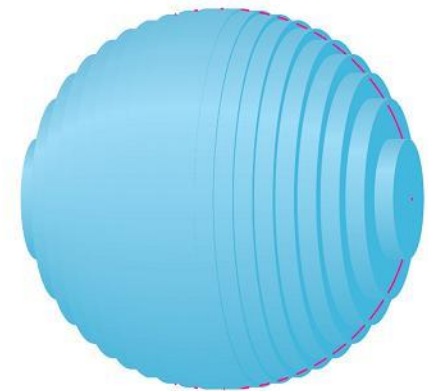Figure 5 illustrates the definition of volume when the solid is a sphere with radius $r = 1$.

From the result of Example 1, we know that the volume of the sphere is $\frac{4}{3}\pi$, which is approximately 4.18879.

(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Figure 5

Approximating the volume of a sphere with radius 1

Volumes

Here the slabs are circular cylinders, or *disks*, and the three parts of Figure 5 show the geometric interpretations of the Riemann sums

$$\sum_{i=1}^{n} A(\bar{x}_i)\, \Delta x = \sum_{i=1}^{n} \pi(1^2 - \bar{x}_i^2)\, \Delta x$$

when $n = 5$, 10, and 20 if we choose the sample points x_i^* to be the midpoints $\bar{x}_i$.

Notice that as we increase the number of approximating cylinders, the corresponding Riemann sums become closer to the true volume.

Volumes

The solids as in Example 1 are all called **solids of revolution** because they are obtained by revolving a region about a line. In general, we calculate the volume of a solid of revolution by using the basic defining formula

$$V = \int_a^b A(x)\, dx \qquad \text{or} \qquad V = \int_c^d A(y)\, dy$$

and we find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

If the cross-section is a disk, we find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

Volumes

If the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch (as in Figure 10) and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

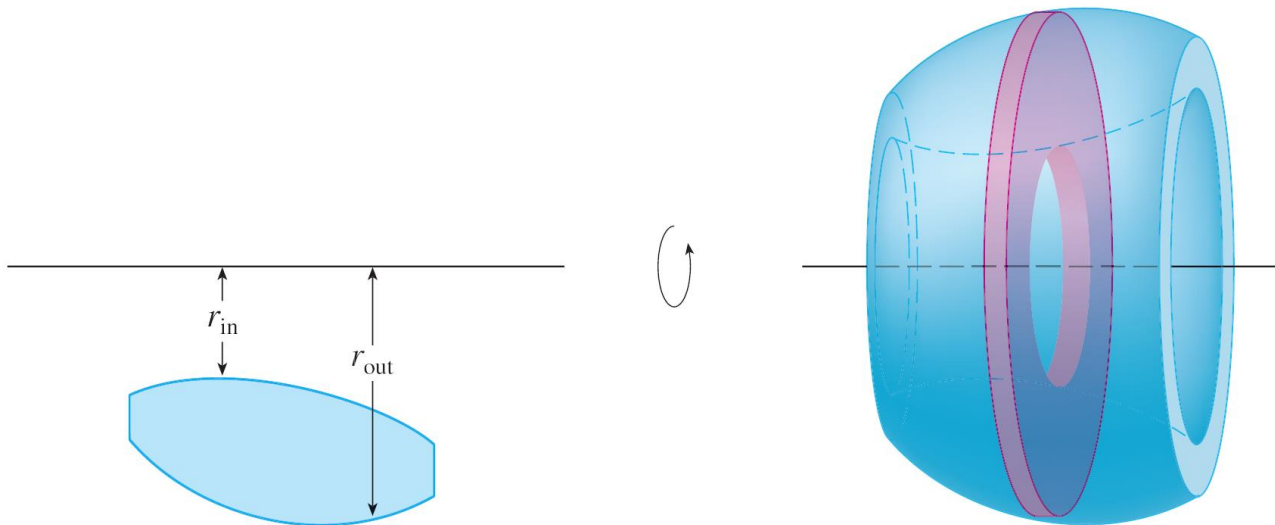$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

Figure 10