Applications of Integration

6

6.3 Volumes by Cylindrical Shells

Volumes by Cylindrical Shells

Let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.)

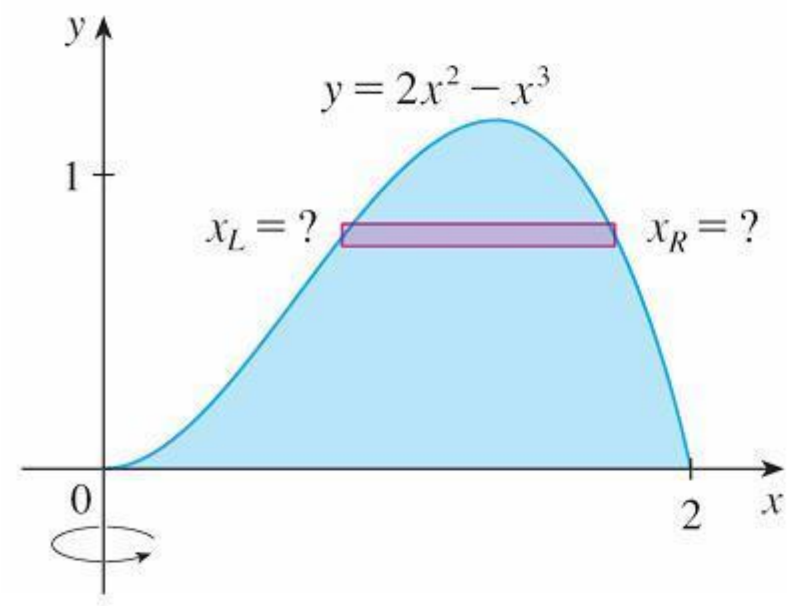

Figure 1

If we slice perpendicular to the y-axis, we get a washer.

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But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h.

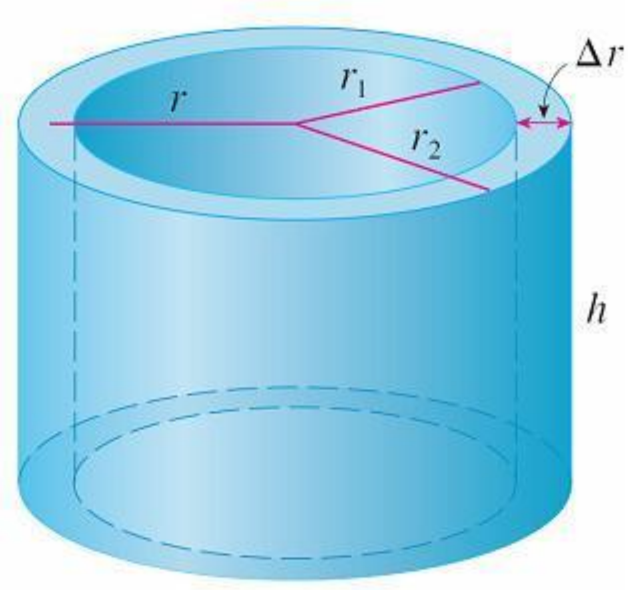

Figure 2

Volumes by Cylindrical Shells

Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1$$

$$= \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h$$

$$= \pi(r_2 + r_1)(r_2 - r_1)h$$

$$= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)$$

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If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

1 $$V = 2\pi r h \, \Delta r$$

and it can be remembered as

V = [circumference] [height] [thickness]

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Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geq 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$. (See Figure 3.)

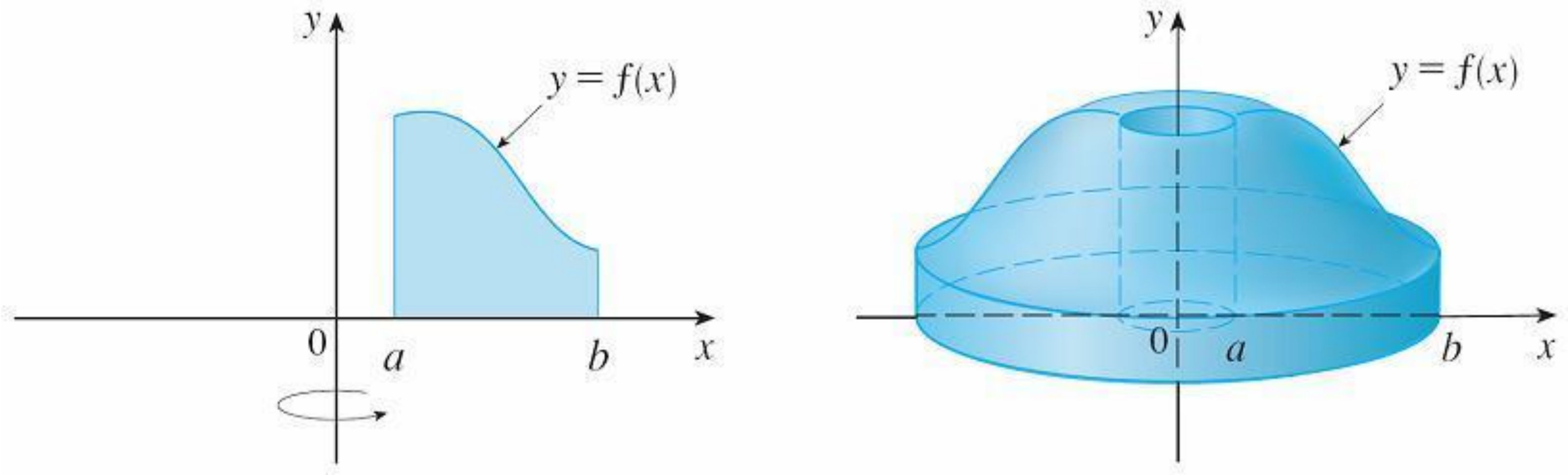

Figure 3

We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval.

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If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx (see Figure 4), so by Formula 1 its volume is

$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

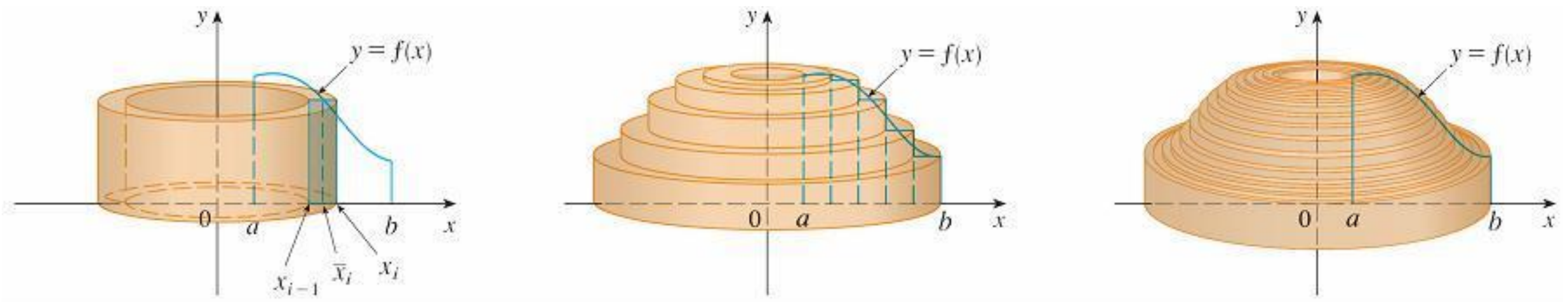

Figure 4

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Therefore an approximation to the volume *V* of *S* is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i)\,\Delta x$$

This approximation appears to become better as $n \to \infty$.

But, from the definition of an integral, we know that

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i)\,\Delta x = \int_a^b 2\pi x f(x)\,dx$$

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Thus the following appears plausible:

2 The volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x)\,dx \qquad \text{where } 0 \leqslant a < b$$

Volumes by Cylindrical Shells

The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in Figure 5, with radius *x*, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \; \underbrace{[f(x)]}_{\text{height}} \; \underbrace{dx}_{\text{thickness}}$$

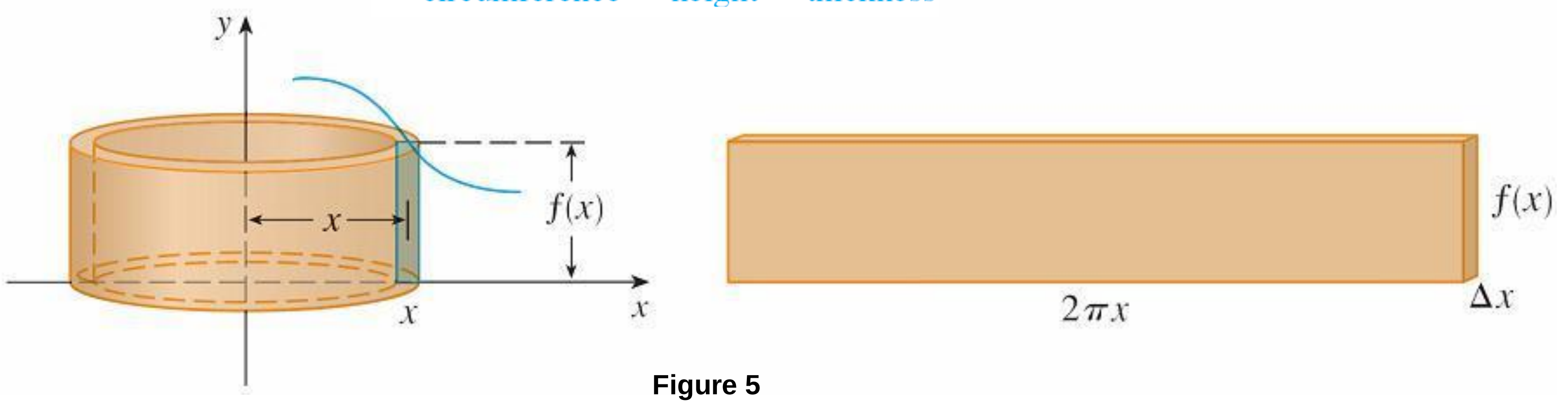

Figure 5

This type of reasoning will be helpful in other situations, such as when we rotate about lines other than the *y*-axis.

Example 1 – *Using the Shell Method*

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Solution:

From the sketch in Figure 6 we see that a typical shell has radius x, circumference $2\pi x$, and height $f(x) = 2x^2 - x^3$.

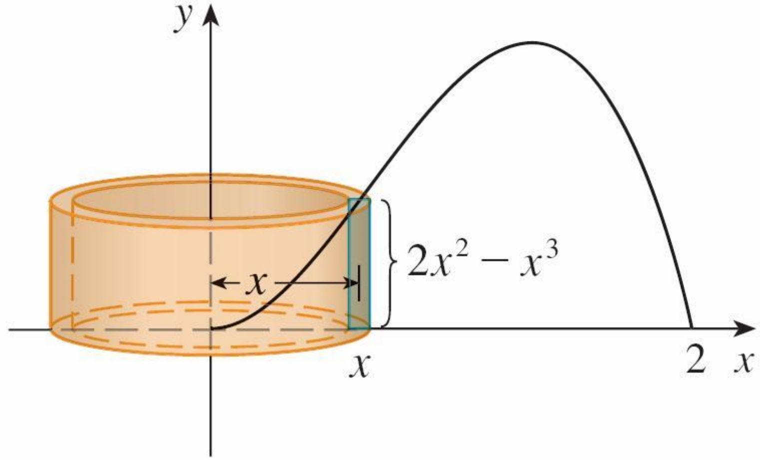

Figure 6

Example 1 – *Solution*

cont'd

So, by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\,dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right)$$

$$= \tfrac{16}{5}\pi$$

It can be verified that the shell method gives the same answer as slicing.