Applications of Integration

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6.4 Arc Length

Arc Length

What do we mean by the length of a curve? We might think of fitting a piece of string to the curve in Figure 1 and then measuring the string against a ruler. But that might be difficult to do with much accuracy if we have a complicated curve.

We need a precise definition for the length of an arc of a curve, in the same spirit as the definitions we developed for the concepts of area and volume.

Figure 1

Arc Length

If the curve is a polygon, we can easily find its length; we just add the lengths of the line segments that form the polygon. (We can use the distance formula to find the distance between the endpoints of each segment.) We are going to define the length of a general curve by first approximating it by a polygon and then taking a limit as the number of segments of the polygon is increased. This process is familiar for the case of a circle, where the circumference is the limit of lengths of inscribed polygons (see Figure 2).

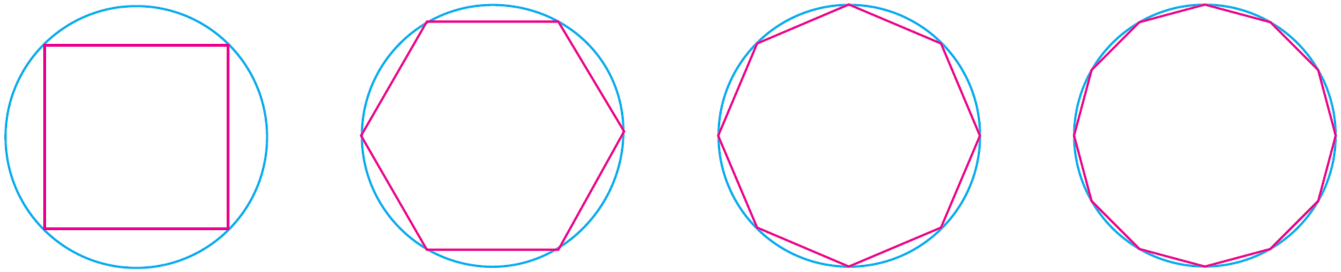

Figure 2

Arc Length

Suppose that a curve C is described by the parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

Let's assume that C is **smooth** in the sense that the derivatives $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero for $a < t < b$. (This ensures that C has no sudden change in direction.)

Arc Length

We divide the parameter interval $[a, b]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots t_n$, are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots P_n$, approximates C. (See Figure 3.)

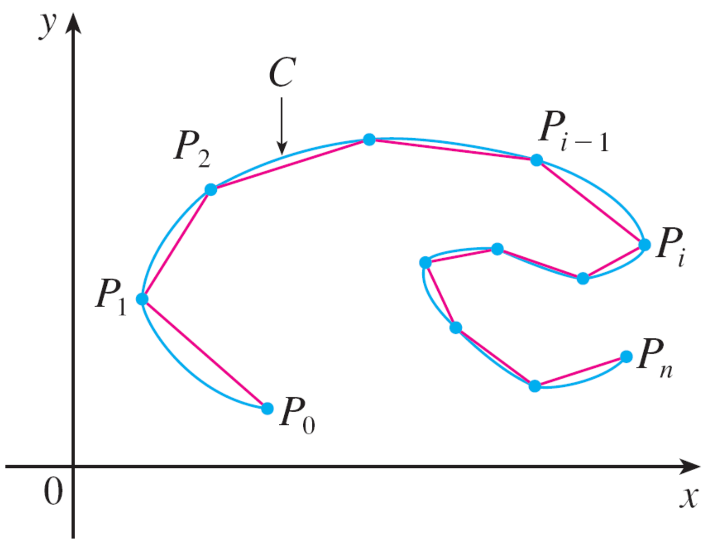

Figure 3

Arc Length

The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. (See Figure 4, where the arc of the curve between P_{i-1} and P_i has been magnified and approximations with successively smaller values of Δt are shown.)

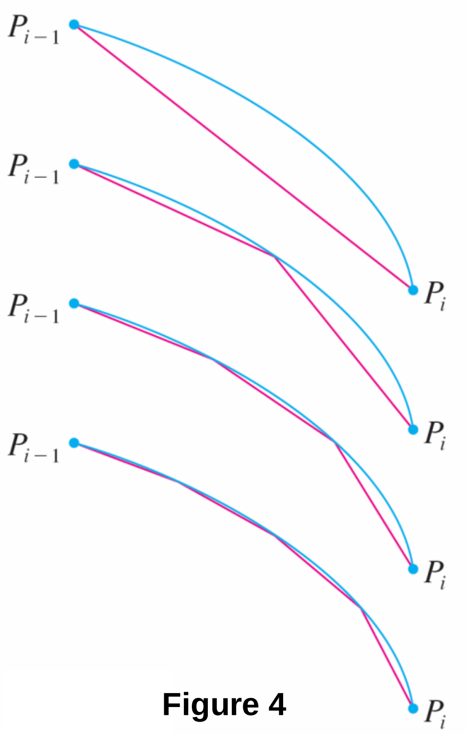

Figure 4

Therefore we define the **length** of C to be the limit of the lengths of these inscribed polygons:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Arc Length

Notice that the procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divided the curve into a large number of small parts. We then found the approximate lengths of the small parts and added them. Finally, we took the limit as $n \to \infty$.

For computational purposes we need a more convenient expression for L. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, then the length of the ith line segment of the polygon is

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Arc Length

But from the definition of a derivative we know that

$$f'(t_i) \approx \frac{\Delta x_i}{\Delta t}$$

if Δt is small. (We could have used any sample point t_i^* in place of t_i.)

Therefore

$$\Delta x_i \approx f'(t_i)\,\Delta t \qquad\qquad \Delta y_i \approx g'(t_i)\,\Delta t$$

and so

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Arc Length

$$\approx \sqrt{[f'(t_i)\,\Delta t]^2 + [g'(t_i)\,\Delta t]^2}$$

$$= \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t$$

Thus

$$L \approx \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t$$

This is a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ and so our argument suggests that

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Arc Length

In fact, our reasoning can be made precise; this formula is correct, provided that we rule out situations where a portion of the curve is traced out more than once.

1 Arc Length Formula If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is traversed exactly once as t increases from a to b, then its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 1 – *Length of a Parametric Curve*

Find the length of the arc of the curve $x = t^2$, $y = t^3$ that lies between the points (1, 1) and (4, 8). (See Figure 5.)

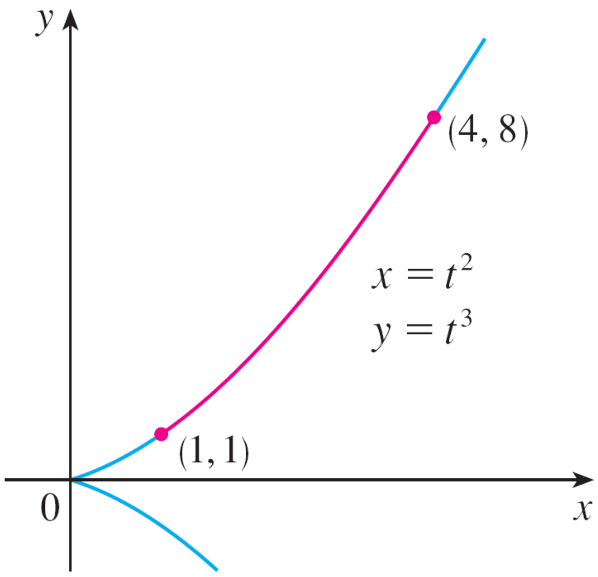

Figure 5

Example 1 – *Solution*

First we notice from the equations $x = t^2$ and $y = t^3$ that the portion of the curve between (1, 1) and (4, 8) corresponds to the parameter interval $1 \le t \le 2$.

So the arc length formula (1) gives

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 t\sqrt{4 + 9t^2}\, dt$$

Example 1 – *Solution*

cont'd

If we substitute $u = 4 + 9t^2$, then $du = 18t\,dt$.

When $t = 1$, $u = 13$; when $t = 2$, $u = 40$.

Therefore

$$L = \frac{1}{18}\int_{13}^{40}\sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big]_{13}^{40}$$

$$= \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right]$$

$$= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$

Arc Length

If we are given a curve with equation $y = f(x)$, $a \le x \le b$, then we can regard x as a parameter. Then parametric equations are $x = x$, $y = f(x)$, and Formula 1 becomes

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$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Similarly, if a curve has the equation $x = f(y)$, $a \le y \le b$, we regard y as the parameter and the length is

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$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

Arc Length

Because of the presence of the root sign in Formulas 1, 2, and 3, the calculation of an arc length often leads to an integral that is very difficult or even impossible to evaluate explicitly.

Thus we often have to be content with finding an approximation to the length of a curve.