Applications of Integration

6

6.5 Average Value of a Function

Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

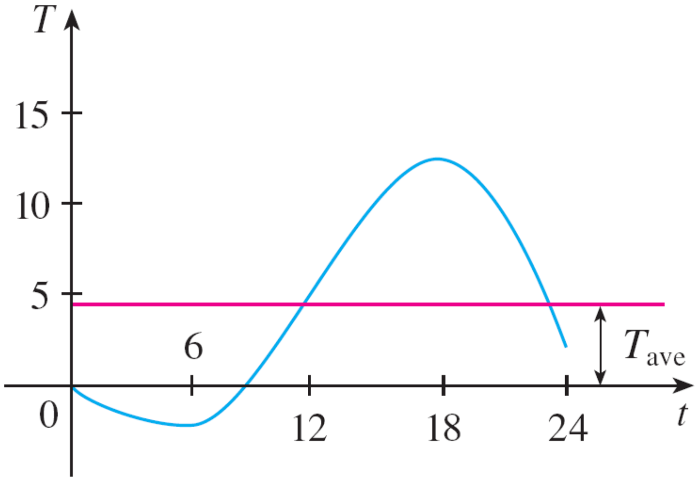

Figure 1

Average Value of a Function

In general, let's try to compute the average value of a function $y = f(x)$, $a \le x \le b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$.

Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers $f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.)

Average Value of a Function

Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values.

Average Value of a Function

The limiting value is

$$\lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [a, b] as

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval [–1, 2].

Solution:

With $a = -1$ and $b = 2$ we have

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} (1 + x^2)\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= 2$$

Average Value of a Function

If $T(t)$ is the temperature at time t, we might wonder if there is a specific time when the temperature is the same as the average temperature.

For the temperature function graphed in Figure 1, we see that there are two such times—just before noon and just before midnight.

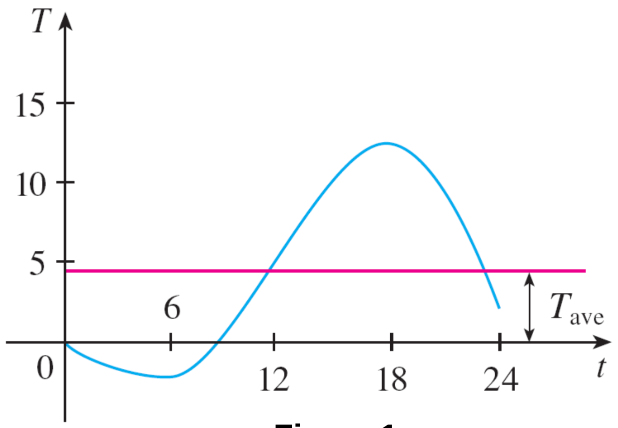

Figure 1

In general, is there a number c at which the value of a function f is exactly equal to the average value of the function, that is, $f(c) = f_{ave}$?

Average Value of a Function

The following theorem says that this is indeed the case for continuous functions.

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus.

Average Value of a Function

The geometric interpretation of the Mean Value Theorem for Integrals is that, for *positive* functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b. (See Figure 2.)

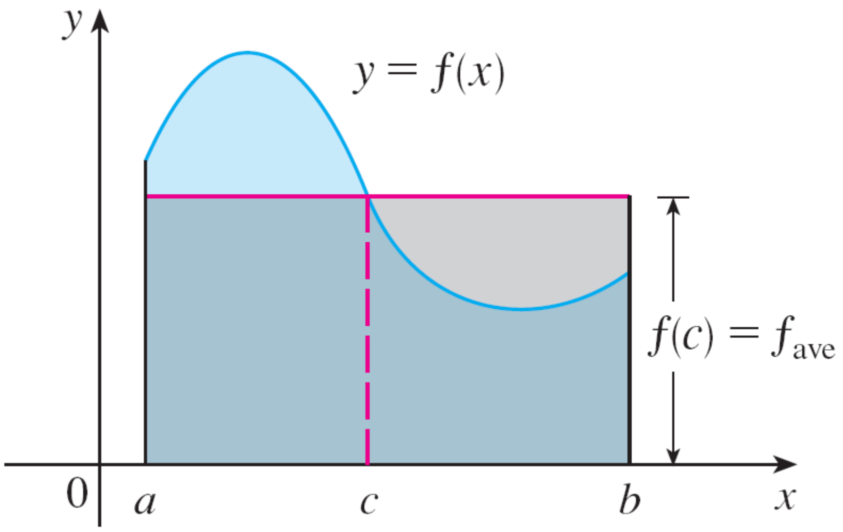

Figure 2