Applications of Integration

6

6.6 Applications to Physics and Engineering

Applications to Physics and Engineering

Among the many applications of integral calculus to physics and engineering, we consider three: work, force due to water pressure, and centers of mass.

Work

Work

The term *work* is used in everyday language to mean the total amount of effort required to perform a task.

In physics it has a technical meaning that depends on the idea of a *force.*

Intuitively, you can think of a force as describing a push or pull on an object—for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball.

Work

In general, if an object moves along a straight line with position function $s(t)$, then the **force** F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration:

1 $$F = m\frac{d^2s}{dt^2}$$

In the SI metric system, the mass is measured in kilograms (kg), the displacement in meters (m), the time in seconds (s), and the force in newtons ($N = kg \cdot m/s^2$).

Work

Thus a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/s^2. In the US Customary system the fundamental unit is chosen to be the unit of force, which is the pound.

In the case of constant acceleration, the force *F* is also constant and the work done is defined to be the product of the force *F* and the distance *d* that the object moves:

2 $$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

If *F* is measured in newtons and *d* in meters, then the unit for *W* is a newton-meter, which is called a joule (J). If *F* is measured in pounds and *d* in feet, then the unit for *W* is a foot-pound (ft-lb), which is about 1.36 J.

Work

For instance, suppose you lift a 1.2-kg book off the floor to put it on a desk that is 0.7 m high. The force you exert is equal and opposite to that exerted by gravity, so Equation 1 gives

$$F = mg = (1.2)(9.8) = 11.76 \text{ N}$$

and then Equation 2 gives the work done as

$$W = Fd = (11.76)(0.7) \approx 8.2 \text{ J}$$

But if a 20-lb weight is lifted 6 ft off the ground, then the force is given as $F = 20$ lb, so the work done is

$$W = Fd = 20 \cdot 6 = 120 \text{ ft-lb}$$

Work

Here we didn't multiply by *g* because we were given the *weight* (a force) and not the mass.

Equation 2 defines work as long as the force is constant, but what happens if the force is variable?

Let's suppose that the object moves along the *x*-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function.

We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.

Work

We choose a sample point x_i^* in the ith subinterval $[x_{i-1}, x_i]$. Then the force at that point is $f(x_i^*)$.

If n is large, then Δx is small, and since f is continuous, the values of f don't change very much over the interval $[x_{i-1}, x_i]$.

In other words, f is almost constant on the interval and so the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately given by Equation 2:

$$W_i \approx f(x_i^*)\, \Delta x$$

Work

Thus we can approximate the total work by

3 $$W \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

It seems that this approximation becomes better as we make *n* larger. Therefore we define the **work done in moving the object from *a* to *b*** as the limit of this quantity as $n \to \infty$. Since the right side of (3) is a Riemann sum, we recognize its limit as being a definite integral and so

4 $$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

Example 1 – *Work Done by a Variable Force*

When a particle is located a distance *x* feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Solution:

$$W = \int_1^3 (x^2 + 2x)\,dx = \frac{x^3}{3} + x^2 \Big]_1^3$$

$$= \frac{50}{3}$$

The work done is $16\frac{2}{3}$ ft-lb.

Work

Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant (called the **spring constant**). Hooke's Law holds provided that x is not too large (see Figure 1).

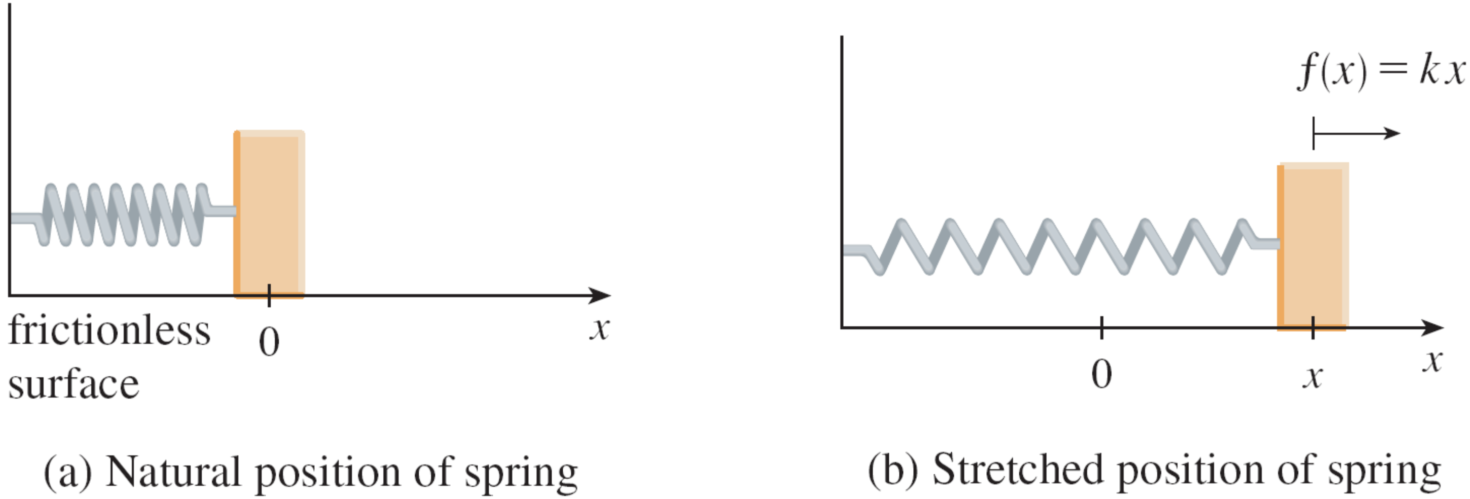

Figure 1 Hooke's Law

Hydrostatic Pressure and Force

Hydrostatic Pressure and Force

Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.

In general, suppose that a thin horizontal plate with area *A* square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth *d* meters below the surface of the fluid as in Figure 5.

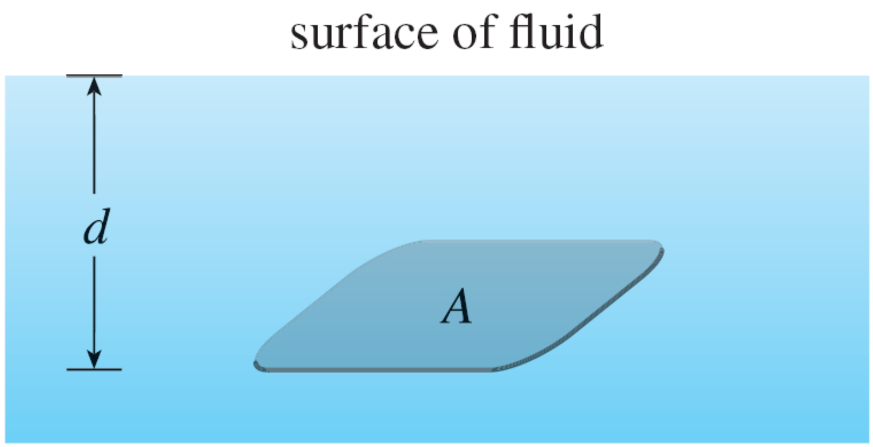

Figure 5

Hydrostatic Pressure and Force

The fluid directly above the plate has volume $V = Ad$, so its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho gAd$$

where g is the acceleration due to gravity. The pressure P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho gd$$

The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation: $1\ \mathrm{N/m^2} = 1\ \mathrm{Pa}$). Since this is a small unit, the kilopascal (kPa) is often used.

Hydrostatic Pressure and Force

For instance, because the density of water is $\rho = 1000\ kg/m^3$, the pressure at the bottom of a swimming pool 2 m deep is

$$P = \rho g d = 1000\ kg/m^3 \times 9.8\ m/s^2 \times 2\ m$$

$$= 19{,}600\ Pa = 19.6\ kPa$$

An important principle of fluid pressure is the experimentally verified fact that *at any point in a liquid the pressure is the same in all directions.* (A diver feels the same pressure on nose and both ears.)

Hydrostatic Pressure and Force

Thus the pressure in *any* direction at a depth d in a fluid with mass density ρ is given by

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$$P = \rho g d = \delta d$$

This helps us determine the hydrostatic force against a *vertical* plate or wall or dam in a fluid.

This is not a straightforward problem because the pressure is not constant but increases as the depth increases.

Example 5 – *Hydrostatic Force on a Dam*

A dam has the shape of the trapezoid shown in Figure 6. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

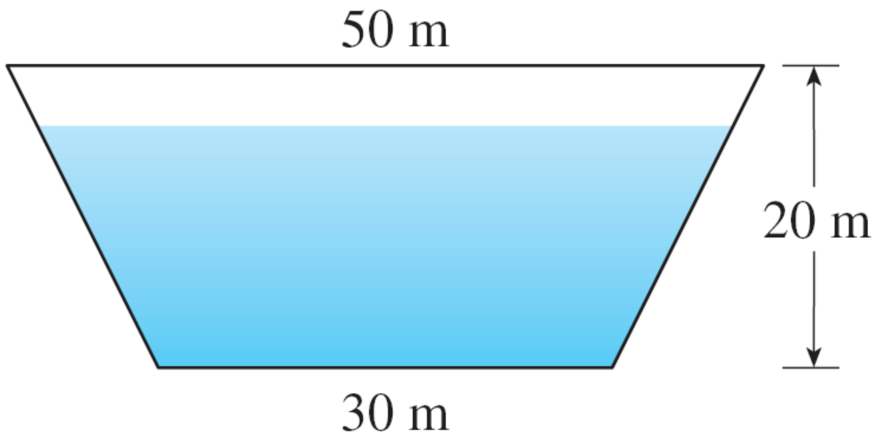

Figure 6

Example 5 – *Solution*

We choose a vertical *x*-axis with origin at the surface of the water as in Figure 7(a).

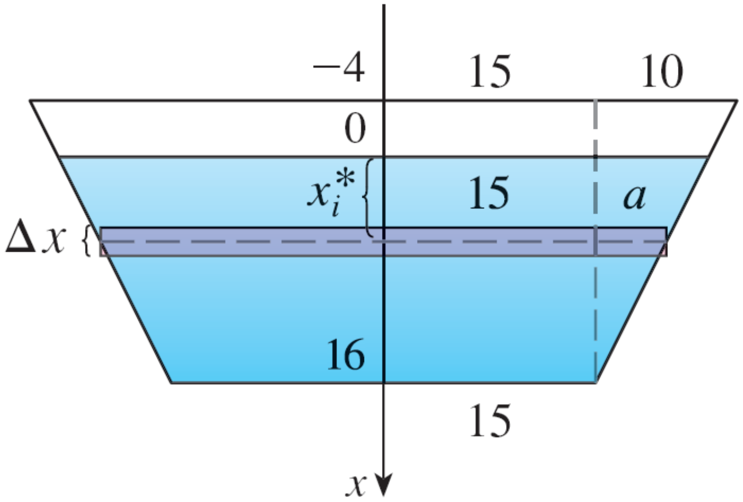

Figure 7(a)

The depth of the water is 16 m, so we divide the interval [0, 16] into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$.

Example 5 – *Solution*

cont'd

The *i*th horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles in Figure 7(b),

$$\frac{a}{16 - x_i^*} = \frac{10}{20}$$

or

$$a = \frac{16 - x_i^*}{2} = 8 - \frac{x_i^*}{2}$$

and so

$$w_i = 2(15 + a)$$

$$= 2(15 + 8 - \tfrac{1}{2}x_i^*)$$

$$= 46 - x_i^*$$

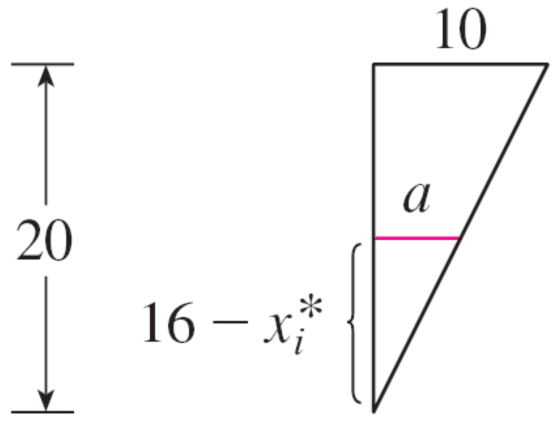

Figure 7(b)

Example 5 – *Solution*

cont'd

If A_i is the area of the ith strip, then

$$A_i \approx w_i \, \Delta x$$
$$= (46 - x_i^*) \, \Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant and we can use Equation 5 to write

$$P_i \approx 1000 g x_i^*$$

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

$$F_i = P_i A_i$$
$$\approx 1000 g x_i^* (46 - x_i^*) \, \Delta x$$

Example 5 – *Solution*

cont'd

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 1000gx_i^*(46 - x_i^*)\,\Delta x$$

$$= \int_0^{16} 1000gx(46 - x)\,dx$$

$$= 1000(9.8) \int_0^{16} (46x - x^2)\,dx$$

$$= 9800\left[23x^2 - \frac{x^3}{3}\right]_0^{16}$$

$$\approx 4.43 \times 10^7 \text{ N}$$

Moments and Centers of Mass

Moments and Centers of Mass

Our main objective here is to find the point *P* on which a thin plate of any given shape balances horizontally as in Figure 8.

This point is called the **center of mass** (or center of gravity) of the plate.

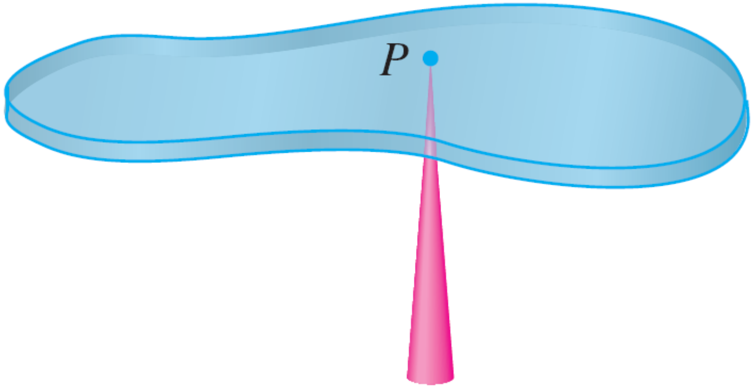

Figure 8

Moments and Centers of Mass

We first consider the simpler situation illustrated in Figure 9, where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

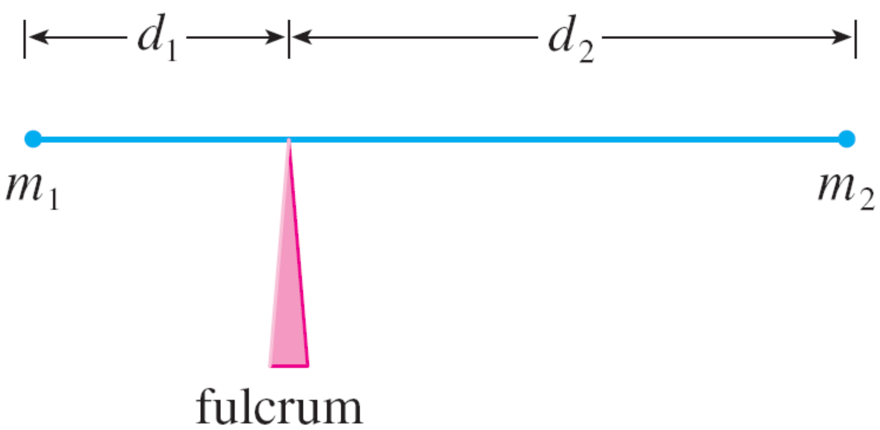

Figure 9

The rod will balance if

$$m_1 d_1 = m_2 d_2 \tag{6}$$

Moments and Centers of Mass

This is an experimental fact discovered by Archimedes and called the Law of the Lever. (Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

Now suppose that the rod lies along the *x*-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\bar{x}$.

Moments and Centers of Mass

If we compare Figures 9 and 10, we see that $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$ and so Equation 6 gives

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

7 $$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

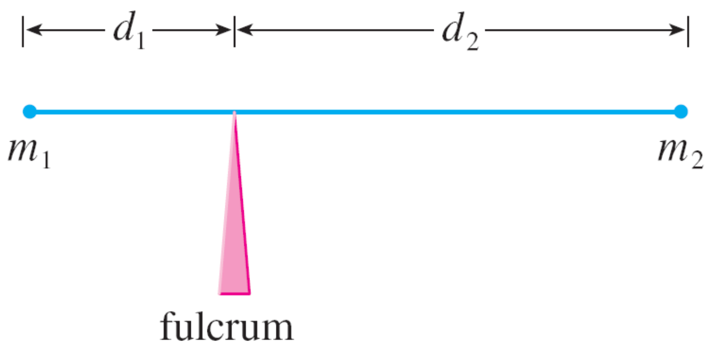

Figure 9

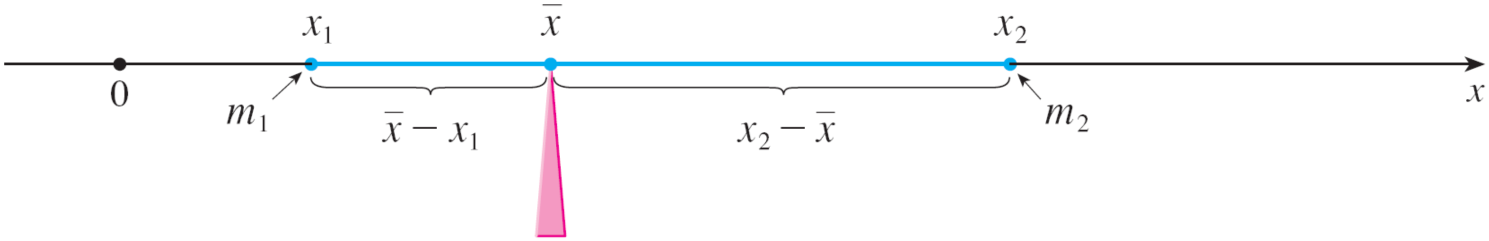

Figure 10

Moments and Centers of Mass

The numbers m_1x_1 and m_2x_2 are called the **moments** of the masses m_1 and m_2 (with respect to the origin), and Equation 7 says that the center of mass $\bar{x}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2$.

Moments and Centers of Mass

In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

8
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \Sigma\, m_i$ is the total mass of the system, and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$

is called the **moment of the system about the origin**.

Moments and Centers of Mass

Then Equation 8 could be rewritten as $m\bar{x} = M$, which says that if the total mass were considered as being concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Now we consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane as shown in Figure 11.

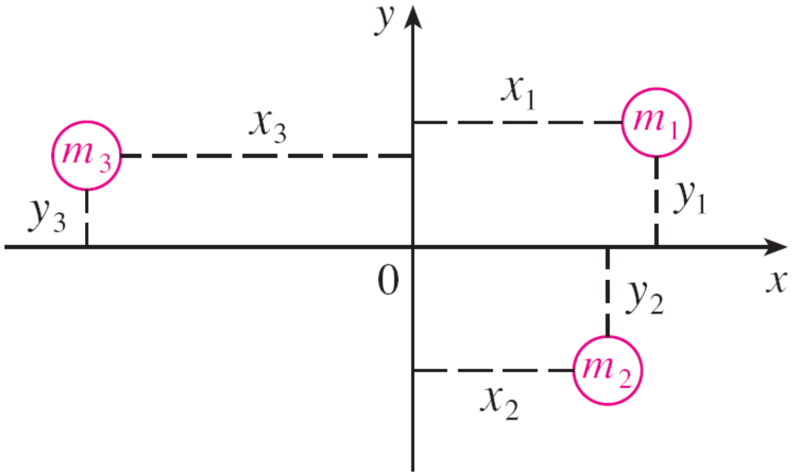

Figure 11

Moments and Centers of Mass

By analogy with the one-dimensional case, we define the **moment of the system about the *y*-axis** to be

9 $$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** as

10 $$M_x = \sum_{i=1}^{n} m_i y_i$$

Then M_y measures the tendency of the system to rotate about the *y*-axis and M_x measures the tendency to rotate about the *x*-axis.

Moments and Centers of Mass

As in the one-dimensional case, the coordinates $(\bar{x}, \bar{y})$ of the center of mass are given in terms of the moments by the formulas

11 $$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

where $m = \Sigma\, m_i$ is the total mass. Since $m\bar{x} = M_y$ and $m\bar{y} = M_x$, the center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

Example 6

Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (–1, 1), (2, –1), and (3, 2).

Solution:

We use Equations 9 and 10 to compute the moments:

$$M_y = 3(-1) + 4(2) + 8(3) = 29$$

$$M_x = 3(1) + 4(-1) + 8(2) = 15$$

Since $m = 3 + 4 + 8 = 15$, we use Equations 11 to obtain

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \qquad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

Example 6 – *Solution*

cont'd

Thus the center of mass is $\left(1\frac{14}{15},\ 1\right)$. (See Figure 12.)

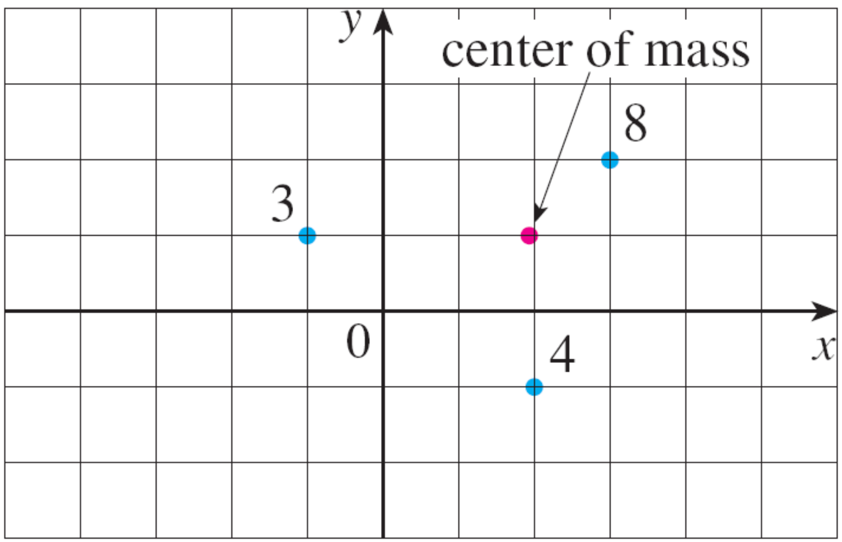

Figure 12

Moments and Centers of Mass

Next we consider a flat plate (called a *lamina*) with uniform density ρ that occupies a region $\mathfrak{R}$ of the plane.

We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathfrak{R}$.

In doing so we use the following physical principles: The **symmetry principle** says that if $\mathfrak{R}$ is symmetric about a line *l*, then the centroid of $\mathfrak{R}$ lies on *l*. (If $\mathfrak{R}$ is reflected about *l*, then $\mathfrak{R}$ remains the same so its centroid remains fixed. But the only fixed points lie on *l*.)

Thus the centroid of a rectangle is its center.

Moments and Centers of Mass

Moments should be defined so that if the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged.

Also, the moment of the union of two nonoverlapping regions should be the sum of the moments of the individual regions.

Suppose that the region $\mathcal{R}$ is of the type shown in Figure 13(a); that is, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the *x*-axis, and beneath the graph of *f*, where *f* is a continuous function.

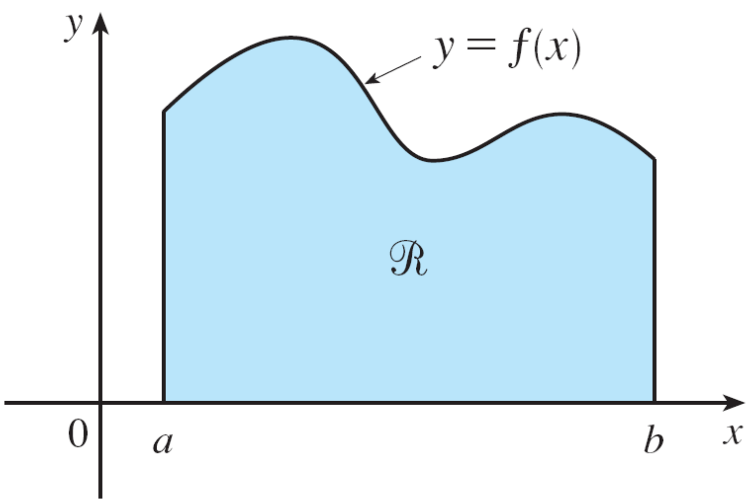

Figure 13(a)

Moments and Centers of Mass

We divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. We choose the sample point x_i^* to be the midpoint $\bar{x}_i$ of the ith subinterval, that is, $\bar{x}_i = (x_{i-1} + x_i)/2$.

This determines the polygonal approximation to $\mathfrak{R}$ shown in Figure 13(b).

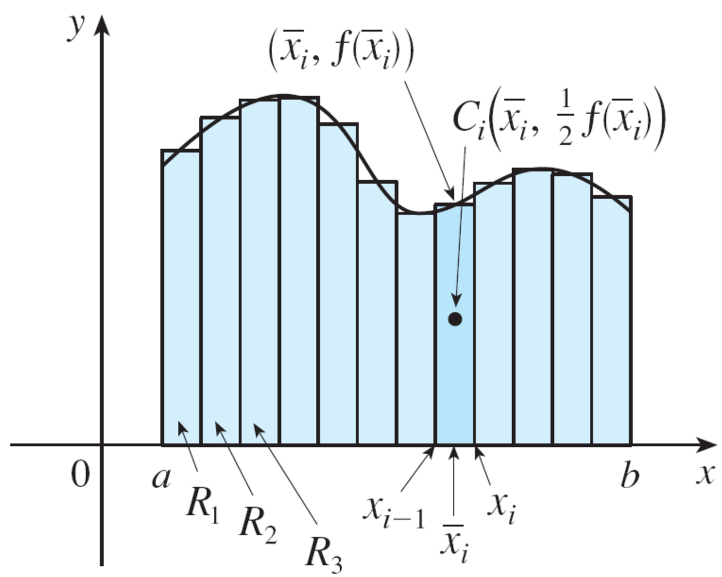

Figure 13(b)

Moments and Centers of Mass

The centroid of the *i*th approximating rectangle R_i is its center $C_i\left(\bar{x}_i, \frac{1}{2}f(\bar{x}_i)\right)$. Its area is $f(\bar{x}_i)\,\Delta x$, so its mass is

$$\rho f(\bar{x}_i)\,\Delta x$$

The moment of R_i about the *y*-axis is the product of its mass and the distance from C_i to the *y*-axis, which is $\bar{x}_i$. Thus

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho\bar{x}_i f(\bar{x}_i)\,\Delta x$$

Adding these moments, we obtain the moment of the polygonal approximation to $\mathcal{R}$, and then by taking the limit as $n \to \infty$ we obtain the moment of $\mathcal{R}$ itself about the *y*-axis:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\bar{x}_i f(\bar{x}_i)\,\Delta x = \rho \int_a^b x f(x)\,dx$$

Moments and Centers of Mass

In a similar fashion we compute the moment of R_i about the x-axis as the product of its mass and the distance from C_i to the x-axis:

$$M_x(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\tfrac{1}{2} f(\bar{x}_i) = \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x$$

Again we add these moments and take the limit to obtain the moment of $\mathfrak{R}$ about the x-axis:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \tfrac{1}{2}[f(\bar{x}_i)]^2\,\Delta x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\,dx$$

Moments and Centers of Mass

Just as for systems of particles, the center of mass of the plate is defined so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. But the mass of the plate is the product of its density and its area:

$$m = \rho A = \rho \int_a^b f(x)\,dx$$

and so

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}$$

Moments and Centers of Mass

Notice the cancellation of the ρ's. The location of the center of mass is independent of the density.

In summary, the center of mass of the plate (or the centroid of $\mathcal{R}$) is located at the point $(\bar{x}, \bar{y})$, where

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$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx \qquad \bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}[f(x)]^2\, dx$$