Applications of Integration

6

6.7 Applications to Economics and Biology

Applications to Economics and Biology

In this section we consider some applications of integration to economics (consumer surplus) and biology (blood flow, cardiac output).

Consumer Surplus

Consumer Surplus

Recall that the demand function $p(x)$ is the price that a company has to charge in order to sell x units of a commodity.

Usually, selling larger quantities requires lowering prices, so the demand function is a decreasing function. The graph of a typical demand function, called a **demand curve**, is shown in Figure 1. If X is the amount of the commodity that is currently available, then $P = p(X)$ is the current selling price.

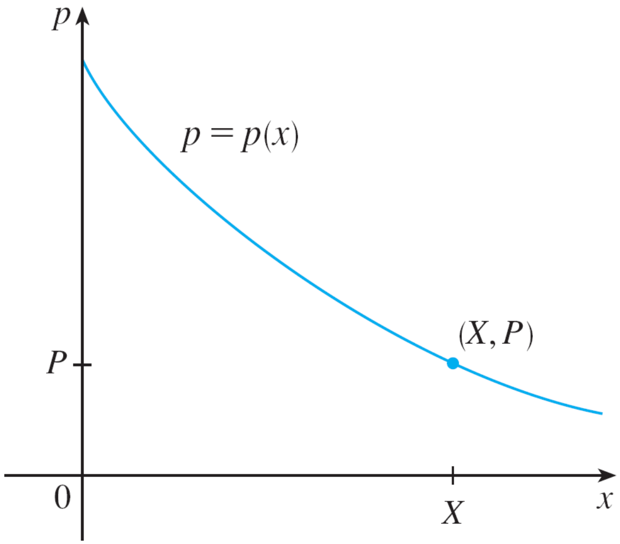

Figure 1

A typical demand curve

Consumer Surplus

We divide the interval [0, X] into n subintervals, each of length $\Delta x = X/n$, and let $x_i^* = x_i$ be the right endpoint of the ith subinterval, as in Figure 2.

If, after the first x_{i-1} units were sold, a total of only x_i units had been available and the price per unit had been set at $p(x_i)$ dollars, then the additional Δx units could have been sold (but no more).

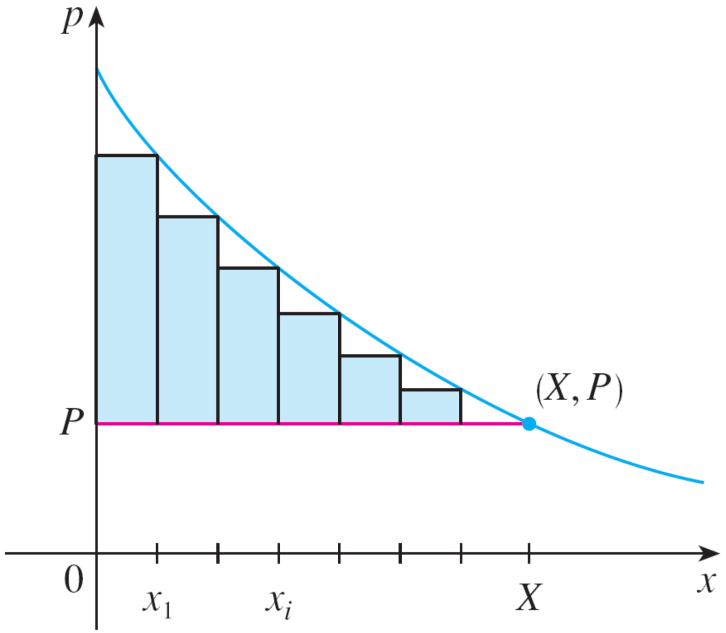

Figure 2

Consumer Surplus

The consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them.

So, in paying only P dollars they have saved an amount of

$$(\text{savings per unit})(\text{number of units}) = [p(x_i) - P]\,\Delta x$$

Consumer Surplus

Considering similar groups of willing consumers for each of the subintervals and adding the savings, we get the total savings:

$$\sum_{i=1}^{n} [p(x_i) - P]\,\Delta x$$

(This sum corresponds to the area enclosed by the rectangles in Figure 2.)

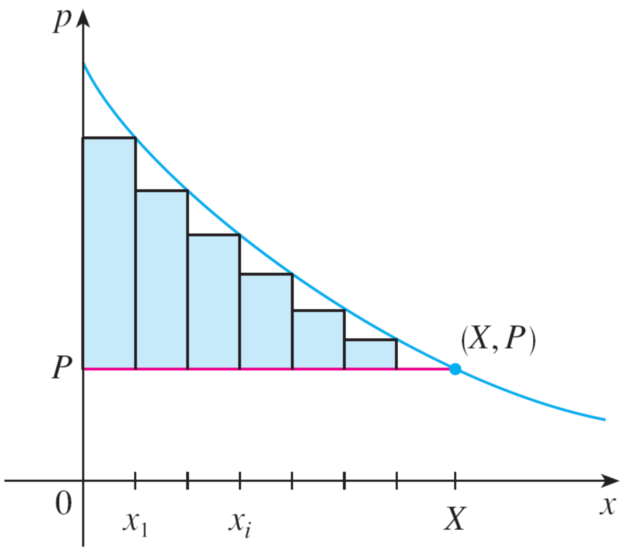

Figure 2

Consumer Surplus

If we let $n \to \infty$, this Riemann sum approaches the integral

$$\boxed{1} \qquad \int_0^X [p(x) - P]\, dx$$

which economists call the **consumer surplus** for the commodity.

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X.

Consumer Surplus

Figure 3 shows the interpretation of the consumer surplus as the area under the demand curve and above the line $p = P$.

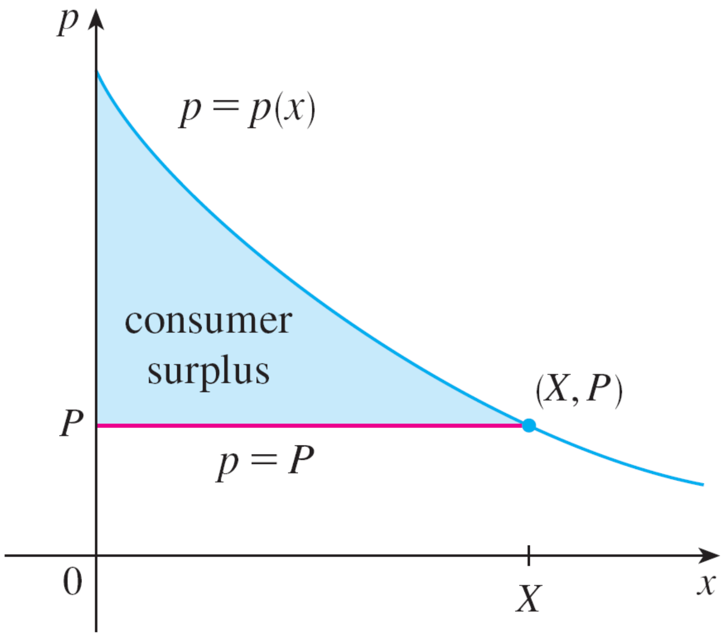

Figure 3

Example 1 – *Consumer Surplus*

The demand for a product, in dollars, is

$$p = 1200 – 0.2x – 0.0001x^2$$

Find the consumer surplus when the sales level is 500.

Solution:

Since the number of products sold is $X = 500$, the corresponding price is

$$P = 1200 – (0.2)(500) – (0.0001)(500)^2$$
$$= 1075$$

Example 1 – *Solution*

cont'd

Therefore, from Definition 1, the consumer surplus is

$$\int_0^{500} [p(x) - P]\, dx = \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075)\, dx$$

$$= \int_0^{500} (125 - 0.2x - 0.0001x^2)\, dx$$

$$= 125x - 0.1x^2 - (0.0001)\left(\frac{x^3}{3}\right)\Bigg]_0^{500}$$

$$= (125)(500) - (0.1)(500)^2 - \frac{(0.0001)(500)^3}{3}$$

$$= \$33{,}333.33$$

Blood Flow

Blood Flow

We have discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity *v* of blood that flows along a blood vessel with radius *R* and length *l* at a distance *r* from the central axis, where *P* is the pressure difference between the ends of the vessel and η is the viscosity of the blood.

Now, in order to compute the rate of blood flow, or *flux* (volume per unit time), we consider smaller, equally spaced radii $r_1, r_2, \ldots$.

Blood Flow

The approximate area of the ring (or washer) with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i \, \Delta r \qquad \text{where} \qquad \Delta r = r_i - r_{i-1}$$

(See Figure 4.)

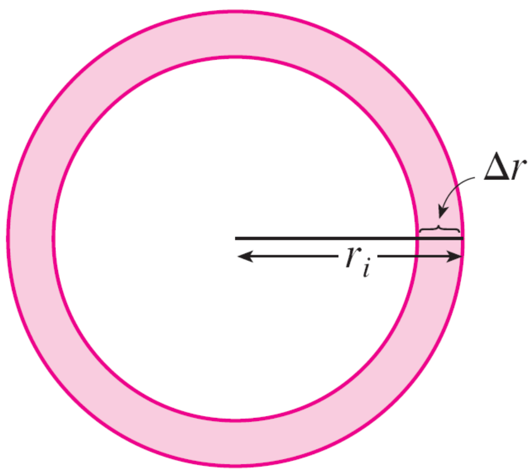

Figure 4

Blood Flow

If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$.

Thus the volume of blood per unit time that flows across the ring is approximately

$$(2\pi r_i \, \Delta r) \, v(r_i) = 2\pi r_i \, v(r_i) \, \Delta r$$

and the total volume of blood that flows across a cross-section per unit time is about

$$\sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r$$

This approximation is illustrated in Figure 5.

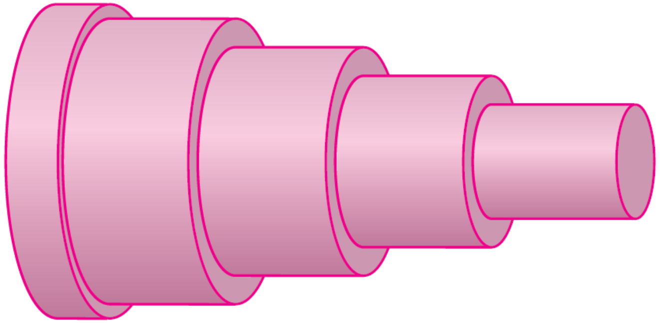

Figure 5

Blood Flow

Notice that the velocity (and hence the volume per unit time) increases toward the center of the blood vessel.

The approximation gets better as *n* increases.

When we take the limit we get the exact value of the **flux** (or *discharge*), which is the volume of blood that passes a cross-section per unit time:

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r_i \, v(r_i) \, \Delta r$$

$$= \int_0^R 2\pi r \, v(r) \, dr$$

Blood Flow

$$= \int_0^R 2\pi r \frac{P}{4\eta l}(R^2 - r^2)\,dr$$

$$= \frac{\pi P}{2\eta l}\int_0^R (R^2 r - r^3)\,dr$$

$$= \frac{\pi P}{2\eta l}\left[R^2 \frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=R}$$

$$= \frac{\pi P}{2\eta l}\left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{\pi P R^4}{8\eta l}$$

Blood Flow

The resulting equation

$$\boxed{2} \qquad F = \frac{\pi P R^4}{8\eta l}$$

is called **Poiseuille's Law**; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

Cardiac Output

Cardiac Output

Figure 6 shows the human cardiovascular system.

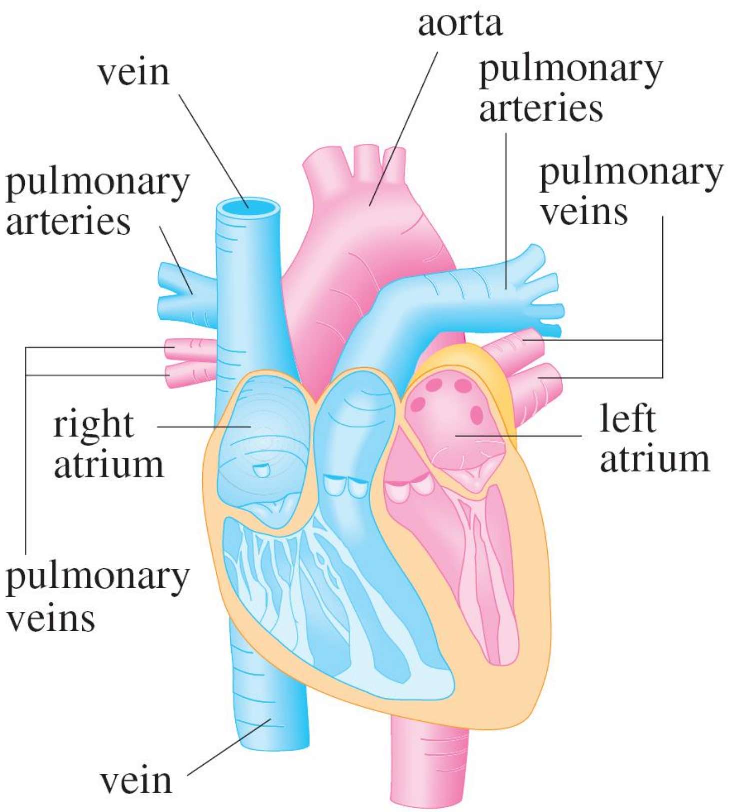

Figure 6

Cardiac Output

Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs through the pulmonary arteries for oxygenation.

It then flows back into the left atrium through the pulmonary veins and then out to the rest of the body through the aorta.

The **cardiac output** of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta.

The *dye dilution method* is used to measure the cardiac output.

Cardiac Output

Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of the dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared.

Let $c(t)$ be the concentration of the dye at time t. If we divide $[0, T]$ into subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is approximately

$$(\text{concentration})(\text{volume}) = c(t_i)(F\,\Delta t)$$

where F is the rate of flow that we are trying to determine.

Cardiac Output

Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i)F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\,\Delta t$$

and, letting n $\rightarrow \infty$, we find that the amount of dye is

$$A = F\int_0^T c(t)\,dt$$

Thus the cardiac output is given by

$$\boxed{3} \qquad F = \frac{A}{\displaystyle\int_0^T c(t)\,dt}$$

where the amount of dye *A* is known and the integral can be approximated from the concentration readings.

Example 2 – *Cardiac Output*

A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

Example 2 – *Solution*

Here A = 5, Δt = 1, and T = 10. We use Simpson's Rule to approximate the integral of the concentration:

$$\int_0^{10} c(t)\,dt \approx \tfrac{1}{3}\,[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)$$

$$+ 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0]$$

$$\approx 41.87$$

Thus Formula 3 gives the cardiac output to be

$$F = \frac{A}{\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87}$$

$$\approx 0.12 \text{ L/s} = 7.2 \text{ L/min}$$